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A New Proof for Irrationality of π

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***Abstract:** Ever since Lambert proved that π is irrational in 18th century, lots of wonderful proofs have been provided by various mathematicians. To this day, π remains as one of the most significant and important real number among all real numbers. In this paper, we try to prove that π is irrational in a new and elementary way. In doing so, we have obtained new rational approximations for π.*

***Keywords:** Irrational, Mediant, Closed Intervals, Rational Approximations*

I. INTRODUCTION

In 1761, Johann Heinrich Lambert was the first to prove that π is irrational using continued fraction corresponding to tangent function. In later years, several proofs of the same were provided by eminent mathematicians like Charles Hermite, Mary Cartwright, Ivan Niven, Miklós Laczkovich. In 1882, Ferdinand Von Lindemann proved that π is not only irrational but it is transcendental as well. In 2006, American mathematician Jonathan Sondow (see [1]) proved that the base of natural logarithm *e* is irrational in most spectacular way using geometric construction. Getting inspired by that proof, in this paper, we will prove that π is irrational in most elegant and easiest possible way. For knowing more about properties of π see [2].

II. MEDIANT OF TWO NUMBERS

If $\frac{a}{b}$ and $\frac{c}{d}$ are two rational numbers then the number $\frac{a+c}{b+d}$ is defined as Mediant of $\frac{a}{b}$and $\frac{c}{d}$. According to the definition, it is fairly straightforward to show that if $\frac{a}{b},\frac{c}{d}$ are positive, then the Mediant always lies between $\frac{a}{b}$and $\frac{c}{d}$. We shall now prove this formally.

A. Theorem 1

The mediant of two positive rational numbers always lies between them. That is, if $\frac{a}{b},\frac{c}{d}$ are positive, and if $\frac{a}{b}<\frac{c}{d}$then $\frac{a}{b}<\frac{a+c}{b+d}<\frac{c}{d}$.

Proof: From $\frac{a}{b}<\frac{c}{d}$ we obtain $bc-ad>0$. $\frac{a+c}{b+d}-\frac{a}{b}=\frac{bc-ad}{b(b+d)}>0$and $\frac{c}{d}-\frac{a+c}{b+d}=\frac{bc-ad}{b(b+d)}>0$. Thus, $\frac{a}{b}<\frac{a+c}{b+d}<\frac{c}{d}$.

III. IRRATIONALITY OF π

In [3], American mathematician Tom Apostol provided a new proof for irrationality of $\sqrt{2}$. In this section, we will prove that π is irrational using the finest weapon that mathematicians use namely Proof by Contradiction method.

A. Theorem 2

The number π is irrational

Proof: If possible, let us assume that π be some rational number. First, we consider the true value of π (from known computations) to few decimal places given by π = 3.14159265358979 323846264338327950…

From above true value [3], we notice that π lies between two consecutive integers 3 and 4. So we will begin our computations from the interval [4]. In particular, let $I_0=[3,4]=\left[\frac{3}{1},\frac{4}{1}\right]$. The mediant of the numbers $\frac{3}{1},\frac{4}{1}$ is $\frac{3+4}{1+1}=\frac{7}{2}=3.5$. By theorem 1, this mediant value must be such that $3<3.5<4$. Now since $3<\pi<3.5$, we consider our next interval as $I_1=[3,3.5]=\left[\frac{6}{2},\frac{7}{2}\right]$. The mediant of $\frac{6}{2},\frac{7}{2}$ is $\frac{6+7}{2+2}=\frac{13}{4}=3.25$. Since $3<\pi<3.25$, we consider $I_2=[3,3.25]=\left[\frac{12}{4},\frac{13}{4}\right]$. Now the mediant of $\frac{12}{4},\frac{13}{4}$ is $\frac{12+13}{4+4}=\frac{25}{8}=3.125$. Since $3.125<\pi<3.25$, we consider $I_3=[3.125,3.25]=\left[\frac{25}{8},\frac{26}{8}\right]$. By our choice of closed intervals at each stage, we notice that $\pi\in I_n$ for $n=0,1,2,3$. We now proceed to determine closed intervals at next stages, such that π is always a number in those intervals.

The mediant of $\frac{25}{8},\frac{26}{8}$ is $\frac{51}{16}=3.1875$. Since $\frac{25}{8}<\pi<\frac{51}{16}$ we choose $I_4=\left[\frac{50}{16},\frac{51}{16}\right]$. The mediant of $\frac{50}{16},\frac{51}{16}$ is $\frac{101}{32}=3.15625$. Since $\frac{25}{8}<\pi<\frac{101}{32}$ we choose $I_5=\left[\frac{100}{32},\frac{101}{32}\right]$. The mediant of $\frac{100}{32},\frac{101}{32}$ is $\frac{201}{64}=3.140625$. Since $\frac{201}{64}<\pi<\frac{101}{32}$ we choose $I_6=\left[\frac{201}{64},\frac{202}{64}\right]$.

Proceeding in same fashion, we obtain sequence of closed intervals given by

$$I_7=\left[\frac{402}{128},\frac{403}{128}\right];\ I_8=\left[\frac{804}{256},\frac{805}{256}\right];$$

$$I_9=\left[\frac{1608}{512},\frac{1609}{512}\right];\ I_{10}=\left[\frac{3216}{1024},\frac{3217}{1024}\right];$$

$$I_{11}=\left[\frac{6433}{2048},\frac{6434}{2048}\right];$$

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$$I_{12} = \left[\frac{12867}{4096}, \frac{12868}{4096}\right]; \; I_{13} = \left[\frac{25735}{8192}, \frac{25736}{8192}\right];$$

$$I_{14} = \left[\frac{51471}{16384}, \frac{51472}{16384}\right]; \; I_{15} = \left[\frac{102943}{32768}, \frac{102944}{32768}\right];$$

$$I_{16} = \left[\frac{205887}{65536}, \frac{205888}{65536}\right]; \; I_{17} = \left[\frac{411774}{131072}, \frac{411775}{131072}\right];$$

$$I_{18} = \left[\frac{823549}{262144}, \frac{823550}{262144}\right]; \; I_{19} = \left[\frac{1647099}{524288}, \frac{1647100}{524288}\right];$$

$$I_{20} = \left[\frac{3294198}{1048576}, \frac{3294199}{1048576}\right];$$

$$I_{21} = \left[\frac{6588397}{2097152}, \frac{6588398}{2097152}\right];$$

$$I_{22} = \left[\frac{13176794}{4194304}, \frac{13176795}{4194304}\right],$$

$$I_{23} = \left[\frac{26353589}{8388608}, \frac{26353590}{8388608}\right];$$

If we keep continuing this process as many times as possible, then we see that $\{\pi\} \in I_n$ for all n = 0, 1, 2, 3, 4 , … Since we select each interval by the choice of mediant of numbers of previous intervals, by theorem 1, we find that $I_0 \supset I_1 \supset I_2 \supset I_3 \supset \cdots$. That is $I_n \supset I_{n+1}$ and $\{\pi\} \in \cap_{n=0}^{\infty} I_n$, where each I_n is a closed interval of the form $I_n = \left[\frac{k}{2^n}, \frac{k+1}{2^n}\right]$ for some natural number k. Also $|I_n| = \frac{1}{2^n} \to 0$ as $n \to \infty$. Therefore by nested interval property of real numbers, $\{\pi\} \in \cap_{n=0}^{\infty} \left[\frac{k}{2^n}, \frac{k+1}{2^n}\right]$, for some natural number k. Since k, k + 1 are consecutive natural numbers and $I_n \supset I_{n+1}$ is a non-increasing sequence of closed intervals such that $|I_n| = \frac{1}{2^n} \to 0$ as $n \to \infty$, there would be no rational number between $\frac{k}{2^n}$ and $\frac{k+1}{2^n}$ whose denominator is 2^n. Thus if π is rational then $\{\pi\} \notin \left[\frac{k}{2^n}, \frac{k+1}{2^n}\right]$ for any natural number k, contradicting the fact that $\{\pi\} \in \cap_{n=0}^{\infty} \left[\frac{k}{2^n}, \frac{k+1}{2^n}\right]$.by our construction. This contradiction proves that π is not rational and therefore π must be irrational.

IV. RATIONAL APPROXIMATIONS FOR π

In proving that π is irrational in theorem 2, we notice that we have obtained new rational approximations for **π.** Those new rational approximations were marked in red color in particular closed intervals in the above proof. Thus, the rational approximations to π for one, two, three, four, five, six, seven decimal places of accuracy to true value of π were given respectively by $\frac{25}{8}, \frac{201}{64}, \frac{3217}{1024}, \frac{51471}{16384}, \frac{411775}{131072}, \frac{1647099}{524288}, \frac{26353589}{8388608}$. If we continue with more stages of computations [4], then we can obtain rational approximations to π with much more desired decimal places of accuracy. In [5], we see the most efficient rational approximations to π and *e* for desired decimal places of accuracy using continued fractions. In [6], the first author has used the Mediant property to obtain the rational approximations [7] for π and *e* [8].

V. CONCLUSION

In this paper, using the concept of mediant of two numbers, we have proved that π is irrational in elementary way. In doing so, we have determined seven rational approximations to π for one to seven decimal places of accuracy. Among these, except the first, other six values were new approximations to π. These new rational approximations would pave way for alternate usage of approximating π. It is also interesting to note that ancient Babylonians used $\frac{25}{8}$ as approximation to which we have obtained here. Also, we note that the most familiar rational approximations namely $\frac{22}{7}, \frac{333}{106}, \frac{355}{113}$ do not appear in our list. These numbers can be obtained using continued fraction method for π. Nevertheless, the ideas presented in this paper would serve as template for proving irrationality of some specific real numbers.

DECLARATION STATEMENT

After aggregating input from all authors, I must verify the accuracy of the following information as the article's author.

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Dr. R. Sivaraman, working as Associate Professor at Dwaraka Doss Goverdhan Doss Vaishnav College, Chennai has more than 25 years of teaching experience at College level. He has been conferred with National Award for Popularizing mathematics among masses in 2016 by Department of Science and Technology, Government of India. He was conferred with Indian National Science Academy (INSA) Teaching award for the year 2018. He has also received State Government Best Science Book Awards in 2011 and 2012. He has provided more than 500 lectures conveying the beauty, applications of Mathematics. He has published more than 300 research papers and had done his Post Doctoral Research Fellowship and Doctor of Science Degree. He has written 42 books in view of popularizing mathematics among common man. He was a member of the Textbook Writing Committee, Tamil Nadu School Education Department for preparing revised mathematics textbook for eleventh class and chairperson for tenth class. He has won more than 75 prestigious awards for his distinguished service to mathematics. He has been taking free classes for college students from very poor background for many years. Propagating the beauty and applications of mathematics to everyone was his life mission.

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