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Analysis of Classical Special Beta & Gamma Functions in Engineering Mathematics and Physics

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Abstract: ***In many areas of applied mathematics, various types of Special functions have become essential tools for Scientists and engineers. Both Beta and Gamma functions are very important in calculus as complex integrals can be moderated into simpler form. In physics and engineering problems require a detailed knowledge of applied mathematics and an understanding of special functions such as gamma and beta functions. The topic of special functions is very important and it is constantly expanding with the existence of new problems in the applied Sciences in this article, we describe the basic theory of gamma and beta functions, their connections with each other and their applicability to engineering problems.to compute and depict scattering amplitude in Reggae trajectories. Our aim is to illustrate the extension of the classical beta function has many uses. It helps in providing new extensions of the beta distribution, providing new extensions of the Gauss hyper geometric functions and confluent hyper geometric function and generating relations, and extension of Riemann-Lowville derivatives. In this Article, we develop some elementary properties of the beta and gamma functions. We give more than one proof for some results. Often, one proof generalizes and others do not. We briefly discuss the finite field analogy of the gamma and beta functions. These are called Gauss and Jacobi sums and are important in number theory. We show how they can be used to prove Fermat's theorem that a prime of the form 4n + 1 is expressible as a sum of two squares. We also treat a simple multidimensional extension of a beta integral,***

Keywords: ***Beta Function, Gamma Function, A Pochhammer symbol in Applied Mathematics, Engineering***

I. INTRODUCTION

A. Contributions and Essential for Understanding

This Research includes the definition and the theory of classical Special Functions. Euler, Gauss Fourier, Bessel, Legendre spent much time on this topic. Besides applied fields such as fluid dynamics, mathematical physics, engineering and other applied Sciences. Special functions have been a wide range of application area in pure mathematics [1]. Knowledge of the properties of gamma and beta functions, which are among the simplest and most important functions, is essential for understanding of many other functions, hyper geometric functions. In recent years, there have been important studies on the extensions of those functions.

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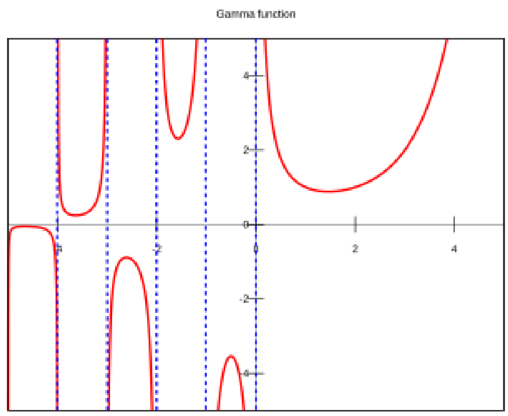
Gamma function is very important in finding solution of Interpolation problem of finding smooth curve

II. THE GAMMA FUNCTION

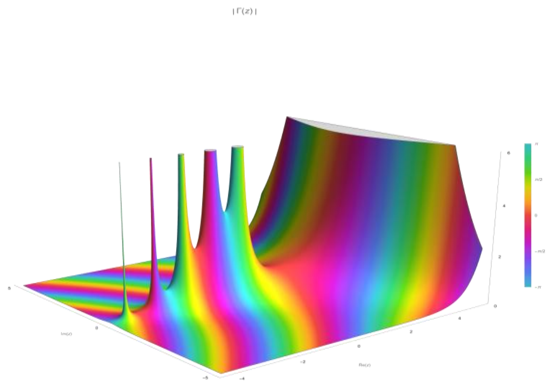
A. The Gamma Function is Defined as Follows

In mathematics, the gamma function (represented by Γ, capital Greek letter gamma) is the most common extension of the factorial function to complex numbers [6]. Derived by Daniel Bernoulli, the gamma function Γ (n). Γ (n) is defined for all complex numbers

The gamma function can be seen as a solution to the interpolation problem of finding a smooth curve y = f(x). y = f(x) that connects the points of the factorial sequences [7].



[Fig.1: Gamma Function [2]]



[Fig.2: Plot of Gamma Function in Complex Plane in 3D [3]]

The Gamma function is defined by the integral formula

$$\Gamma(n) = \int_0^\infty x^{n-1} e^{-x} dx \quad \ldots \quad (1)$$

Where Re (n) ≥ 0 and n € C. This formula was found by Euler (1729) and the notation Γ (n) was introduced by Legendre (18140. The Literature on the gamma function consists of thousands of pages and includes almost 300 years of researches in English, Latin, German and other Languages [8].

The gamma function provides

$$\Gamma\,(\mathrm{n}+1) = n\int_0^\infty x^{n-1}\,e^{-x}\mathrm{dx} = \mathrm{n}\,\Gamma\,(\mathrm{n}) \ldots \quad (2)$$

The recurrence relation. This relation is called the Euler's functional equation. Discovered by Euler in 1729 and this quality gives us the basic property of factorial [9].
From equation (2) we can write

$$\Gamma\,(\mathrm{n}+1) = \mathrm{n}\,(\mathrm{n}-1)\,\Gamma\,(\mathrm{n}-1)$$
$$\Gamma\,(\mathrm{n}+1) = \mathrm{n}\,(\mathrm{n}-1)\,(\mathrm{n}-2)\,\Gamma\,(\mathrm{n}-2)$$

If we continue in this way, from Γ (1) = 1 then we reach the following results:

$$\Gamma\,(\mathrm{n}+1) = \mathrm{n}! \ \ldots \quad (3)$$

This result is called the Euler's functional equation. Which was discovered by Euler in 1729?
However, the gamma function we have defined for positive values of n.

(-1 < n <0) Γ (n) can be found since Γ (n+1) is known.

$$\Gamma\,(\mathrm{n}) = \frac{\Gamma\,(\mathrm{n}+1)}{n} \quad (0 < \mathrm{n}+1 < 1)$$

(-2 < n <-1) Γ (n) can be found since Γ (n+2) is known.

$$\Gamma\,(\mathrm{n}) = \frac{\Gamma\,(\mathrm{n}+2)}{n\,(n+1)} \quad (0 < \mathrm{n}+2 < 1)$$

(-3 < n < -2) Γ (n) can be found since Γ (n+3) is known.

$$\Gamma\,(\mathrm{n}) = \frac{\Gamma\,(\mathrm{n}+3)}{n(n+1)(n+2)} \quad (0 < \mathrm{n}+3 < 1)$$

Similarly for (-n < n < -n+1) Γ (n) can be found since Γ (n+ r) is known

$$\Gamma\,(\mathrm{n}) = \frac{\Gamma\,(\mathrm{n}+\mathrm{r})}{n(n+1)(n+2)---(n+r-1)} \quad (0 < \mathrm{n}+\mathrm{r} < 1)$$

The above equations show that gamma function is unbounded for zero and negative integers and it is finite for all other values of n

The Gamma Function serves as a super powerful version of the Factorial function extending it beyond whole numbers!

The factorial function (symbol :!) says to multiply the whole numbers (from our chosen number down to 1.

Examples: 4! = 4x3x2x1 = 24, 7! = 7x6x5x4x3x2x1 = 5040

III. BEYOND WHOLE NUMBERS

Factorials work beautifully for whole numbers, but what if we want things like 1.2! Or 3.5!
Can we have a function that works more generally?
If so, what properties do we want?
First, it should "hit the mark" at each whole number
F (n) = n F (n−1) in between the whole numbers
F (2.62) = 2.62 × F (1.62)

A. Gamma Distribution [4]

The gamma distribution is one of the continuous distributions; the distributions are very versatile and give useful presentations of many physical situations.

Example:

The burn out time of most products especially electrical devices usually has the gamma Distribution.

To illustrate the application of gamma distribution to the burn out time of products, we shall

Take a sample of your light bulbs with time at 3.9, 4.1, 4.5, and 5.0 days. We shall estimate the parameters using the method of moments of the above gamma distribution

$$\Gamma\,(\mathrm{n}) = \frac{\lambda(\lambda \mathrm{x})\alpha-1\,e^{-\lambda x}}{\mathrm{F}\,(\alpha)},$$ x>0 hence we shall obtain the mean E(X) and Variable (x) get

α = 4.375λ λ = 24.7315

T

Theorem 2.1: $\Gamma\,(\frac{1}{2}) = \sqrt{\pi}$ ... (4)

This property of the gamma function was found by Euler

Theorem 2.2: $\Gamma\,(\mathrm{n})\Gamma\,(1-\mathrm{n}) = \frac{\pi}{\sin(\pi x)}$ ... (5)

Where Re (n) > 0 and n € C. This formula is called Euler's completion formula

Theorem 2.3: $\Gamma\,(2\mathrm{n})\Gamma\left(\frac{1}{2}\right) = 2\;2\mathrm{n}-1\,\Gamma\,(\mathrm{n})\;\Gamma\left(n+\frac{1}{2}\right)$ ... (6)

Where n € C/Z g,¯. This formula is called Legendre's duplication formula

Theorem 2.4: if Re (a) > 0, Re (b) > 0 and a € C and b € C then we write

$$\int_0^{\pi/2} dt \quad \mathrm{Cos\;a}-1\mathrm{t\;sins\;b}-1\mathrm{t} = \frac{1}{2}\,\frac{\left(\Gamma\left(\frac{a}{2}\right)\right)\left(\Gamma\left(\frac{a}{2}\right)\right)}{\Gamma\,\frac{a+b}{2}} \quad \ldots \quad (7)$$

This property was first defined by Whittaker

B. The Pochhammer Symbol [5]

Super symmetry and String Theory by Michael Dine

Definition 3.1:

If Z € R or Z€C and r is zero or a positive integer then we write

$$(\mathrm{Z}) = \mathrm{Z}\,(\mathrm{Z}+1)(\mathrm{Z}+2)\ldots\ldots\ldots\ldots(\mathrm{Z}+\mathrm{r}-1 \quad \ldots \quad (8)$$

The above expression is known as the Pochhammer symbol and it was first defined by Pochhammer in1870.
From the known properties of gamma function, the following features of Pochhammer symbol can be written

$$\Gamma\,(z+r) = (\mathrm{z}+\mathrm{r}-1)\,\Gamma\,(z+r-1)$$
$$\Gamma\,(z+r) = (\mathrm{z}+\mathrm{r}-1)\,(\mathrm{z}+\mathrm{r}-2)\,\Gamma\,(z+r-2)$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$\Gamma\,(z+r) = (\mathrm{z+r-1})(\mathrm{z+r-2})\ldots\ldots\ldots\ldots(\mathrm{z+1})\,\mathrm{z}\,\Gamma\,(z)$$
$$\Gamma\,(z+r) = (\mathrm{z})_{\mathrm{r}}\,\Gamma\,(z)$$

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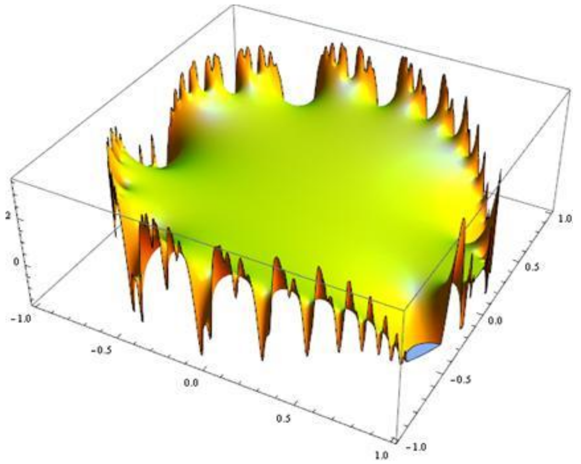
$$(z)r = \frac{\Gamma(z+r)}{\Gamma(z)} \quad \ldots \quad (9)$$

And

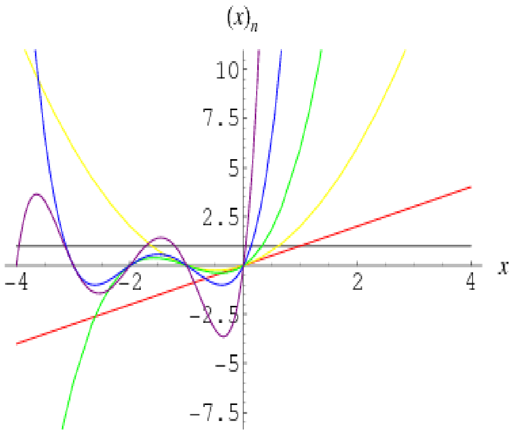
$$(z)_{r+1} = \frac{\Gamma(z+r+1)}{\Gamma(z)} = \frac{z\,\Gamma(z+r+1)}{z\Gamma(z)} = z\,\frac{\Gamma((z+r)+1)}{\Gamma(z+1)} =$$

$Z(Z+1)_r$ ... (10)

Specially, if we take r = 0 in equation (9), it is seen that$(Z)_0 = 1$



[Fig.3: Pochhammer Symbol in 3D [2]]



[Fig.4: Graph of Pochhammer [2]]

IV. THE BETA FUNCTION

Definition 4.1:

The Beta function is defined as follows

$$B(m, n) = = \int_0^1 x^{m-1}\,(1-x)^{n-1}dx, \quad \ldots \quad (11)$$

Where if Re (a) > 0, Re (b) > 0 and a € C and b € C This formula was found by Euler

If we define a new integration variable x = k/(1+k). Then (11) becomes

$$B(m,n) = \int_0^\infty \frac{k^{a-1}}{(1+k)\;a+b}\,dK\;k^{a-1} \quad \ldots \quad (12)$$

This expression of beta function in terms of the gamma function is as follows:

$$\frac{\Gamma((m)\Gamma((n)}{\Gamma(m+n)} = B(m,n) \quad \ldots \quad (13)$$

If we choose m+ n = 1 in equation (12)

B (m. 1-m**)** =

$$\int_0^\infty \frac{k^{a-1}}{(1+k)}\,dK = \frac{\pi}{\sin(\pi m)} \qquad 0 < m < 1$$

$$= k^{a-1}$$

Using the above property, we can see that $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$ such that for m = $\frac{1}{2}$

$$B\left(\tfrac{1}{2}, 1\text{-}\tfrac{1}{2}\right) = \frac{\Gamma\left(\left(\frac{1}{2}\right)\Gamma\left(\left(\frac{1}{2}\right)\right.\right.}{\Gamma\left(\frac{1}{2}+\frac{1}{2}\right)} = B\left(\tfrac{1}{2}, \tfrac{1}{2}\right) = \frac{\pi}{\sin(\pi/2)} = \pi,$$

$$\left(\Gamma\left(\left(\tfrac{1}{2}\right)\right)^2 = \pi,\right.$$

$$\Gamma\left(\left(\tfrac{1}{2}\right)\right) = \sqrt{\pi}$$

Also beta function is symmetric to its variables

$$\frac{\Gamma((m)\Gamma((n)}{\Gamma(m+n)} = B(n, m) = \frac{\Gamma((n)\Gamma((m)}{\Gamma(n+m)} = B(m,n) \quad \ldots \quad (14)$$

Theorem 4.1:

$$B(m, n) = = \int_0^\infty x^{m-1}\,(1-x)^{-(m+n)}\,dx,$$

And B (m, n) = B (m+1, n) + B (m, n+1)

These results were found by Watson and Whittaker

V. CONCLUSION

In this paper, we introduced some important and fundamental properties and the theory of Gamma function, Pochammer Symbols, Beta function and we discussed the relation with other definitions. We believe that these analyses will be useful for researchers to understand the theory of gamma and beta functions. We can construct a beautiful visualization of hyper geometric series plots.

DECLARATION STATEMENT

I must verify the accuracy of the following information as the article's author.

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- **Authors Contributions:** The authorship of this article is contributed solely.

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7. *National Conference on Advancements* of *Science, Technology, Engineering, and Management* Analysis of infinitesimal Calculus and Development of ability to solve problems (IJSRED).

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