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On Laplace-Riemann Derivatives

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***Abstract:** The basic properties of the n^{th} Laplace-Riemann derivative of a function f at a point x is studied.*

***Keywords:** Laplace-Riemann Derivatives, Riemann - Derivatives, Derivative, Continuity, Mathematical Subject Classification 2020: 26E99, 28E99.*

I. INTRODUCTION

In the past different types of derivatives were defined and investigated [1]. Laplace derivative and Riemann derivative are such two derivatives [6]. Laplace derivative was first introduced in and studied in [2]. Laplace-Riemann derivative is another generalization of ordinary derivative which is defined with the help of the concept of the previous two derivatives [5]. In this section, we have studied the order Laplace-Riemann derivative and have shown by example that the Laplace-Riemann derivative is more general than the ordinary derivative [3]. Also, we have proved some theorems regarding monotonicity and Mean value property for the Laplace-Riemann derivative of a function having Upper semi-continuity and property D [8].

A. Definitions and Notations

Definition 1.1: Let $f:\mathbf{R}\to\mathbf{R}$ be a function, which is specially Denjoy-integrable in some neighborhood of $x\in R$ .If the limit

$$\lim_{s\to\infty}\frac{s^{n+1}}{n!}\int_0^{\delta}e^{-st}\Delta^n(f,x,t)dt$$

exists then it is said to be the n^{th} right Laplace-Riemann derivative of f at x and is denoted by $LRD_n^+f(x)$. If the limit

$$\lim_{s\to\infty}(-1)^n\frac{s^{n+1}}{n!}\int_0^{\delta}e^{-st}\Delta^n(f,x,-t)dt$$

exists then it is said to be the left Laplace-Riemann derivative of order n off at x and is denoted by $LRD_n^-f(x)$.

If both $LRD_n^+f(x)$ and $LRD_n^-f(x)$ exist and are equal, then the common value is called the $n-th$ Laplace-Riemann derivative of f at x and is denoted by $LRD_nf(x)$ [4].

The Definitions are independent of δ [4].

B. Propeties of Laplace-Riemann Derivative

To study properties of Laplace-Riemann derivative, following lemmas are used.

Lemma 1.1 : If $\psi(t)=o(t^n)$, then

(i)

$$\lim_{s\to\infty}\frac{s^{n+1}}{n!}\int_0^{\delta}e^{-st}\psi(t)dt=0,(ii)\lim_{h\to 0}\frac{1}{h}\int_0^{h}\frac{\psi(t)}{t^n}dt=0.$$

The proof is given in [4].

Lemma 1.2: If p, q are positive integers and $\delta>0$ then

$$s^q\int_0^{\delta}e^{-st}t^pdt=p!s^{q-p-1}+o(1)\text{ as }s\to\infty.$$

The proof is given in [5]

Lemma 1.3: If p, q are positive integers and $\delta>0$ then

$$s^q\int_0^{\delta}e^{-st}t^pdt=p!s^{q-p-1}+o(1)\text{ as }s\to\infty.$$

The proof is clear [6]

Also, we know:

1.For $f:\mathbf{R}\to\mathbf{R}$ be a function, which is special Denjoy integrable in some neighborhood of $x\in R$, if the n^{th} Peano derivative of f at x i. e. $f_n(x)$ exists then $LRD_nf(x)$ exists and $f_n(x)=LRD_nf(x)$ .The converse of the Theorem is not true [4].

2.If the n^{th} general derivative of f at x i. e. $f^n(x)$ exists then the $n-th$ Laplace-Riemann derivative of f at x i. e. $LRD_nf(x)$ exists with same value but not conversely [3].

3. $RD_n^-f(x)\le LRD_n^-f(x)\le LRD_n^+f(x)\le RD_n^+f(x)$

Here, $RD_n^+f(x)$ and $RD_n^-f(x)$ are the right and left n^{th} Riemann derivative off at x respectively. The converse is not true in general [4].

4. $LD_n^-f(x)\le LRD_n^-f(x)\le LRD_n^+f(x)\le LD_n^+f(x)$

Here, $LD_n^+f(x)$ and $LD_n^-f(x)$ are the right and left n^{th} Laplace derivative off at x respectively. The converse is not true in general [7].

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Remark: Laplace-Riemann derivative is more general than ordinary derivative, Peano derivative, Laplace derivative, Riemann derivative.

C. Basic Property

1) $LRD_n(f+g)(x) = LRD_n f(x) + LRD_n g(x)$.

Proof.

$$LRD_n(f+g)(x)$$
$$=\lim_{s\to\infty}\frac{s^{n+1}}{n!}\int_0^{\delta} e^{-st}\Delta^n(f+g,x,t)dt$$
$$=\lim_{s\to\infty}\frac{s^{n+1}}{n!}\int_0^{\delta} e^{-st}\sum_{j=0}^{n}(-1)^{n-j}\binom{n}{j}(f+g)(x+jt)dt$$
$$=\lim_{s\to\infty}\frac{s^{n+1}}{n!}\int_0^{\delta} e^{-st}\sum_{j=0}^{n}(-1)^{n-j}\binom{n}{j}f(x+jt)dt+\lim_{s\to\infty}\frac{s^{n+1}}{n!}\int_0^{\delta} e^{-st}\sum_{j=0}^{n}(-1)^{n-j}\binom{n}{j}g(x+jt)dt$$
$$=\lim_{s\to\infty}\frac{s^{n+1}}{n!}\int_0^{\delta} e^{-st}\Delta^n(f,x,t)dt+\lim_{s\to\infty}\frac{s^{n+1}}{n!}\int_0^{\delta} e^{-st}\Delta^n(g,x,t)dt$$
$$=LRD_n f(x)+LRD_n g(x).$$

2)For a scalar k, $LRD_n(kf)(x) = kLRD_n f(x)$

Proof.

$$LRD_n(kf)(x)$$
$$=\lim_{s\to\infty}\frac{s^{n+1}}{n!}\int_0^{\delta} e^{-st}\Delta^n(kf,x,t)dt$$
$$=\lim_{s\to\infty}\frac{s^{n+1}}{n!}\int_0^{\delta} e^{-st}\sum_{j=0}^{n}(-1)^{n-j}\binom{n}{j}(kf)(x+jt)dt.$$
$$=k\lim_{s\to\infty}\frac{s^{n+1}}{n!}\int_0^{\delta} e^{-st}\Delta^n(f,x,t)dt$$
$$=kLRD_n f(x)$$

3) $LRD_n(f-g)(x) = LRD_n f(x) - LRD_n g(x)$.

[Evident from 1) and 2)]

D. Example of Laplace-Riemann Derivative of Some Common Functions

(i)Let $f(x)=e^x$.

$$\Delta^1(f,x,t)=f(x+t)-f(x)=e^{x+t}-e^x=[e^t-1]e^x$$

$$\lim_{s\to\infty}s^2\int_0^{\delta}e^{-st}\Delta^1(f,x,t)dt=\lim_{s\to\infty}s^2\int_0^{\delta}e^{-st}[e^t-1]e^xdt=\lim_{s\to\infty}s^2e^x\left[\frac{e^{-(s-1)\delta}-1}{1-s}-\frac{e^{-s\delta}-1}{-s}\right]=e^x$$

Thus, $LRD_1 f(x)=e^x$

$$\Delta^2(f,x,t)=f(x+2t)-f(x+t)+f(x)=e^{x+2t}-e^{x+t}+e^x=[e^{2t}-e^t+1]e^x$$
$$LRD_2 f(x)$$
$$=\lim_{s\to\infty}\frac{s^3}{2!}\int_0^{\delta}e^{-st}\Delta^2(f,x,t)dt$$
$$=\lim_{s\to\infty}\frac{s^3}{2!}\int_0^{\delta}e^{-st}[e^{2t}-e^t+1]e^xdt$$

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$$=e^{x}\lim_{s\to\infty}\frac{s^{3}}{2!}\left[\frac{e^{-(s-2)\delta}-1}{2-s}-\frac{e^{-(s-1)\delta}-1}{1-s}+\frac{e^{-s\delta}-1}{-s}\right]$$

$$=e^{x}\lim_{s\to\infty}\frac{s^{3}}{2!}\left[\frac{e^{-(s-2)\delta}}{2-s}-\frac{e^{-(s-1)\delta}}{1-s}+\frac{e^{-s\delta}}{-s}\right]+e^{x}\lim_{s\to\infty}\frac{s^{3}}{2!}\left[\frac{-1}{2-s}-\frac{-1}{1-s}+\frac{-1}{-s}\right]$$

$$=0+e^{x}$$

$$=e^{x}$$

Thus $LRD_2 f(x)=e^{x}$

(ii)Let $f(x)=x^{2}$.

$$\Delta(f,x,t)=f(x+t)-f(x)=(x+t)^{2}-x^{2}=t(2x+t)=2xt+t^{2}$$

$$\Delta^{2}(f,x,t)$$
$$=\Delta f(x+t)-\Delta f(x)$$
$$=[f(x+2t)-f(x+t)]-[f(x+t)-f(x)]$$
$$=f(x+2t)-2f(x+t)+f(x)$$
$$=(x+2t)^{2}-2(x+t)^{2}+x^{2}$$
$$=2t^{2}$$

$$LRD_1 f(x)=\lim_{s\to\infty}\frac{s^{2}}{1!}\int_0^{\delta}e^{-st}\Delta^{1}(f,x,t)dt=2x\lim_{s\to\infty}\frac{s^{2}}{1!}\int_0^{\delta}e^{-st}tdt+2\lim_{s\to\infty}\frac{s^{2}}{1!}\int_0^{\delta}e^{-st}t^{2}dt=2x$$

$$LRD_2 f(x)=\lim_{s\to\infty}\frac{s^{3}}{2!}\int_0^{\delta}e^{-st}\Delta^{2}(f,x,t)dt=2\lim_{s\to\infty}\frac{s^{3}}{2!}\int_0^{\delta}e^{-st}t^{2}dt=2.$$

II. MAIN RESULTS

Theorem 2.1: Let f be a non-decreasing function in [a, b], then $LRD_1^{+}f\geq 0$ in [a,b]. The converse is also true.

Proof. Let δ be arbitrary small number such that $x+\delta\in[a,b]$ whenever $x\in[a,b]$.

$$LRD_1^{+}f(x)=\lim_{s\to\infty}s^{2}\int_0^{\delta}e^{-st}\Delta(f,x,h)dh=\lim_{s\to\infty}s^{2}\int_0^{\delta}e^{-st}[f(x+h)-f(x)]dh\geq 0,$$

as $f(x+h)-f(x)\geq 0$ for all $h\in[0,\delta]$. Hence, $LRD_1^{+}f\geq 0$ in [a,b].

Conversely, suppose $LRD_{1_1}^{+}f\geq 0$ in [a,b]

$$.So,\lim_{s\to\infty}s^{2}\int_0^{\delta}e^{-st}[f(x+h)-f(x)]dh\geq 0$$

$$\Rightarrow f(x+h)-f(x)\geq 0 \text{ for all } h\in[0,\delta].$$

Therefore, f is non-decreasing in [a, b]

Theorem 2.2 : Let f be a function which is continuous in [a,b], $LRD_1^{+}f$ and $LRD_1^{-}f$ exist in a set E contained in [a, b], then $LRD_1^{+}f, LRD_1^{-}f\in\mathbf{B_1(E)}$. Moreover, if (i) $LRD_p f$ is finite, (ii) $LRD_i f$ is continuous in E, $i=0,1,...,p$, (iii) $LRD_{p+1}^{+}f$ and $LRD_{p+1}^{-}f$ exist in E, then $LRD_{p+1}^{+}f, LRD_{p+1}^{-}f\in\mathbf{B_1(E)}$.

Proof.

Given f is a function which is continuous in [a, b], $LRD_1^{+}f$ and $LRD_1^{-}f$ exist in a set E contained in [a, b].

Since $LRD_1^{+}f$ and $LRD_1^{-}f$ exist in E,

$$\lim_{s\to\infty}s^{2}\int_0^{\delta}e^{-st}\Delta(f,x,t)dt$$ and

$$\lim_{s\to\infty}(-1)^{n}s^{2}\int_0^{\delta}e^{-st}\Delta(f,x,t)dt$$

exist in E. Let,

$$F_n(x) = n^2\int_0^{\delta} e^{-nt}\Delta^1(f,x,t)dt, G_n(x) = (-1)n^2\int_0^{\delta} e^{-nt}\Delta^1(f,x,t)dt$$

It is obvious that $F_n(x), G_n(x)$ are continuous in E.

$$\lim_{n\to\infty} F_n(x) = LRD_1^+ f(x), \lim_{n\to\infty} G_n(x) = LRD_1^- f(x)$$

So, $LRD_1^+ f(x), LRD_1^- f(x) \in \mathbf{B_1(E)}$.

Suppose, moreover, if (i) $LRD_p f$ is finite, (ii) $LRD_i f$ is continuous in E, $i = 0,1,\dots,p$, (iii) $LRD_{p+1}^+ f$ and $LRD_{p+1}^- f$ exist in E. Let,

$$Phi_n(x) = \frac{n^{p+2}}{(p+1)!}\int_0^{\delta} e^{-nt}\Delta^{p+1}(f,x,t)dt, \Psi_n(x) = (-1)^{p+1}\frac{n^{p+2}}{(p+1)!}\int_0^{\delta} e^{-nt}\Delta^{p+1}(f,x,t)dt,$$

It is obvious that $\Phi_n(x), \Psi_n(x)$ are continuous in E.

$$n\lim_{s\to\infty}\Phi_n(x) = LRD_{p+1}^+ f(x), \lim_{n\to\infty}\Psi_{p+1}(x) = LRD_{p+1}^- f(x)$$

So, $LRD_{p+1}^+ f(x), LRD_{p+1}^- f(x) \in \mathbf{B_1(E)}$.

Note 2.1. : Let f be a function in [a, b]. If f is non-decreasing in [a, b], then $LRD_n f(x) \ge 0$ in [a,b].

Proof. Suppose $\alpha, \beta \in [a,b]$, such that $\alpha < \beta$? So, $f(\alpha) \le f(\beta)$.

Now, for any $x_0 \in (a,b)$ and for any δ satisfying $0 < \delta < (b - x_0)$, we have $f(x_0 + \delta) \ge f(x_0)$.

$$\Delta^n(f,x,h) = \sum_{i=0}^{n}(-1)^{n-i}\binom{n}{i} f(x+ih)$$

Let us take $h(>0)$ in a way such that $max\{0,h,2h,\dots,(n-1)h\} \le \delta$.

Hence,

$$\Delta^n(f,x_0,h)$$
$$= \sum_{i=0}^{n}(-1)^{n-i}\binom{n}{i} f(x_0+ih)$$
$$\Rightarrow \Delta^n(f,x_0,h) \ge \sum_{i=0}^{n-1}(-1)^{n-i}\binom{n}{i} f(x_0) + f(x_0+nh)$$
$$\Rightarrow \Delta^n(f,x_0,h) \ge f(x_0)\sum_{i=0}^{n}(-1)^{n-i}\binom{n}{i} + f(x_0+nh) - f(x_0)$$
$$\Rightarrow \Delta^n(f,x_0,h) \ge f(x_0)(-1+1)^n + f(x_0+nh) - f(x_0)$$
$$\Rightarrow \Delta^n(f,x_0,h) \ge f(x_0+nh) - f(x_0) \ge 0$$

Then

$$LRD_n f(x) = \lim_{s\to\infty}\frac{s^{n+1}}{n!}\int_0^{\delta} e^{-st}\Delta^n(f,x,h)dh \ge 0,$$

provided the limit exists.

Theorem 2.3: Let f be an upper semi-continuous function with the property D in the closed interval [a,b]. If $E = \{x \in [a,b] : LRD_n^+ f(x) f \le 0\}$ and $f(E)$ has no sub-interval, then f is non-decreasing in [a,b].

Proof. Suppose $\alpha, \beta \in [a,b]$, such that $\alpha < \beta$. So, $f(\alpha) > f(\beta)$.

Now, let $y_0 \in (f(\alpha), f(\beta))$ such that y_0 doesn't belong to $f(E)$.

Let $S = \{x \in [a,b] : f(x) \ge y_0\}$ and $x_0 = supS$.

Since f is an upper semi-continuous function with property D in [a,b], S is closed and thus $x_0 \in S$. Therefore, $f(x_0) \ge y_0$. We will show that $f(x_0) = y_0$.

If not, there exist η satisfying $f(\beta) < y_0 < \eta < f(x_0)$ and $\xi \in (x_0, \beta)$, such that $f(\xi) = \eta$. It contradicts that $x_0 = supS$. So, $f(x_0) = y_0$.

Since f is an upper semi-continuous function with property D in [a,b] and $x_0 < \beta$, for $x_0 < x < \beta$, $f(x) < f(x_0)$.

If $0 < \delta < (\beta - x_0)$, then $f(x_0 + \delta) - f(x_0) < 0$.

Again, f being upper semi-continuous function with property D in [a, b], for any $y_0 > y$ there is a neighbourhood U of x_0 such that $y < f(x) < y_0$, whenever $x \in U$.

$$\Delta^n(f,x_0,h) = \sum_{i=0}^{n}(-1)^{n-i}\binom{n}{i} f(x_0+ih)$$

Let us take $h(>0)$ in a way such that $x_0 + ih \in U_{x_0+nh}$ for all $i = 0,1,\dots,n$ and $max\{0,h,2h,\dots,(n-1)h\} \le \delta$.

Therefore,

$$\Delta^n(f,x_0,h)$$
$$= \sum_{i=0}^{n}(-1)^{n-i}\binom{n}{i} f(x_0+ih)$$
$$\Rightarrow \Delta^n(f,x_0,h) < \sum_{i=0}^{n-1}(-1)^{n-i}\binom{n}{i} f(x_0) + f(x_0+nh)$$
$$\Rightarrow \Delta^n(f,x_0,h) < f(x_0)\sum_{i=0}^{n}(-1)^{n-i}\binom{n}{i} + f(x_0+nh) - f(x_0)$$
$$\Rightarrow \Delta^n(f,x_0,h) < f(x_0)(-1+1)^n + f(x_0+nh) - f(x_0)$$
$$\Rightarrow \Delta^n(f,x_0,h) < f(x_0+nh) - f(x_0) < 0$$

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Then

$$LRD_n f(x_0) = \lim_{s\to\infty} \frac{s^{n+1}}{n!} \int_0^{\delta} e^{-st} \Delta^n (f, x_0, h) dh \leq 0,$$

implies $x_0 \in S$ and hence $y_0 \in E$, a contradiction.

So, our initial assumption is wrong. There cannot be $\alpha, \beta \in [a,b]$, such that $\alpha < \beta$. So, $f(\alpha) > f(\beta)$. So, f is non-decreasing in [a, b].

Theorem 2.4 : Let f be an upper semi-continuous function which has the property D in [a, b], $LRD_n f(x) \geq 0$ in [a,b] except an enumerable set E. Then f is non-decreasing in [a, b].

Proof. Suppose $\grave{o} > 0$ be arbitrarily small number and $g(x) = f(x) + \grave{o}x$.

$$\begin{aligned} &LRD_n^+ g(x) \\ &= \lim_{s\to\infty} \frac{s^{n+1}}{n!} \int_0^{\delta} e^{-st} \Delta^n (g, x, h) dh \\ &= \lim_{s\to\infty} \frac{s^{n+1}}{n!} \int_0^{\delta} e^{-st} \Delta^n (f, x, h) dh + \grave{o} \lim_{s\to\infty} \frac{s^{n+1}}{n!} \int_0^{\delta} e^{-st} \Delta^n (I, x, h) dh, I(x) = x \\ &= LRD_n f(x), \end{aligned}$$

as $\Delta^n (I, x, h) = 0$

Here, g is also an upper semi-continuous function with property D in [a, b], moreover $g(E)$ is measurable thus contains no sub-interval. So, g is non-decreasing in [a,b].

Since $\grave{o}$ is arbitrarily small positive number, f is non-decreasing in [a, b].

Theorem 2.5 : Let f be an upper semi-continuous function which has the property D in [a, b], $LRD_n f(x) \geq 0$ in [a,b] almost everywhere in [a,b], $LRD_n^+ f(x) > -\infty$ in [a, b] except an enumerable set E. Then f is non-decreasing in [a,b]

Proof. Let $A = \{x \in [a,b] : LRD_n^+ f(x) < 0\}$. Clearly, $m(A) = 0$. Suppose σ is a continuous, non-decreasing function in [a,b] such that $\Delta^n (\sigma, x, h) \geq 0$ in [a,b] except A.

We consider an arbitrary small positive number $\grave{o}$ and take $g(x) = f(x) + \grave{o}\sigma(x)$. Then g an upper semi-continuous function with property D in [a,b],

$$\begin{aligned} &LRD_n^+ g(x) \\ &= \lim_{s\to\infty} \frac{s^{n+1}}{n!} \int_0^{\delta} e^{-st} \Delta^n (g, x, h) dh \\ &= \lim_{s\to\infty} \frac{s^{n+1}}{n!} \int_0^{\delta} e^{-st} \Delta^n (f, x, h) dh + \grave{o} \lim_{s\to\infty} \frac{s^{n+1}}{n!} \int_0^{\delta} e^{-st} \Delta^n (\sigma, x, h) dh \\ &= LRD_n f(x) + \grave{o} RD_n \sigma(x), \end{aligned}$$

Therefore, $LRD_n^+ g(x) \geq 0$ almost everywhere in [a, b] except A. Hence, g is non-decreasing in [a, b].

Since $\grave{o}$ is arbitrarily small positive number, f is non-decreasing in [a, b].

Note 2.1: Example of function σ which is continuous, non-decreasing in [a, b] such that $\Delta^n (\sigma, x, h) \geq 0$ in [a, b] except a set A of measure zero is polynomial $ax^k + bx^{k-2} + \ldots + \lambda$, where the co-efficients are all positive and k is an even natural number.

Theorem 2.6: If f is continuous and $LRD_n f(x)$ exists then $LRD_n^+ f(x)$ has Darboux property.

Proof. Let us consider that $LRD_n^+ f(x)$ does not have Darboux property, then there exist α, β such that $f(\alpha) < 0, f(\beta) > 0$ but $LRD_n^+ f(x) \neq 0$ for any $x \in (\alpha, \beta)$.

Further, suppose $E^+ = \{x \in [\alpha, \beta] : LRD_n^+ f(x) > 0\}, E^- = \{x \in [\alpha, \beta] : LRD_n^+ f(x) < 0\}$, then $[\alpha, \beta] = E^+ \bigcup E^-$.

Let $\mathbf{Q}$ be (if any) non-degenerate component of E^+. Then $\mathbf{Q}$ is an interval. Suppose c, d be the end points of $\mathbf{Q}$.

$LRD_n^+ f > 0$ in $\mathbf{Q}$, so f is non-decreasing in $\mathbf{Q}$. f being continuous and non-decreasing in [c,d], $LRD_n^+ f(c), LRD_n^+ f(d) > 0$. Therefore $c, d \in \mathbf{Q}$, implies that $\mathbf{Q}$ is a closed interval. $\mathbf{Q}$ being arbitrary, every non-degenerate component of E^+ is a closed interval. Following similar arguments, it can be shown that every non-degenerate component of E^- is a closed interval.

Let Q^+, Q^- be the collection of all non-degenerate components of E^+ and E^- respectively. Let $\mathbf{Q}' = Q^+ \bigcup Q^-$. Then any two distinct members of $\mathbf{Q}'$ are disjoint.

Hence, $P = [\alpha, \beta] - \bigcup Q^0, Q \in \mathbf{Q}'$, is perfect and $LRD_n^+ f$ has no point of continuity in P relative to P, which is a contradiction as $LRD_n^+ f \in B[\alpha, \beta]$.

Therefore, $LRD_n^+ f(x)$ must have Darboux property.

Theorem 2.7: Mean Value Theorem

If f is continuous in [a,b] and $LRD_n f(x)$ exists in (a,b) then there exists $c \in (a,b)$ such that $f(b) - f(a) = (b-a) LRD_n f(c)$.

Proof. Here we may have following two cases -

Case 1: Let $f(b) = f(a)$.

Then,

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Subcase 1- In case $LRD_n f(x) \geq 0$ or $LRD_n f(x) \leq 0$ in (a,b). Thus f is monotone function. Now f being continuous as well as monotone, f is constant in (a, b), ensuring the existence of c.

Subcase 2- In case f is not monotone, $LRD_n f(\alpha) < 0$ and $LRD_n f(\beta) > 0$ for some α, β in (a, b) and hence there exists $\xi \in (a,b)$ such that $LRD_n f(\xi) = 0?$, implying $c = \xi$.

Case 2: Let $f(b) \neq f(a)$. Then, suppose $\Phi(x) = f(x) - Ax$, A =constant. Clearly, Φ is continuous in [a, b] and $LRD_n \Phi(x)$ exists in (a, b).

Also, $LRD_n \Phi(x) = LRD_n f(x)$.

Let us take $A = \dfrac{f(b) - f(a)}{b - a}$. Thus, $\Phi(b) = \Phi(a)$.

By Case 1, there exists $c \in (a,b)$ such that $LRD_n \Phi(c) = 0$

$$\Rightarrow LRD_n f(c) = \frac{f(b) - f(a)}{b - a}.$$

This completes the proof of the theorem.

III. CONCLUSION

If f is continuous in [a, b], $f(a) = f(b)$ and $LRD_n f(x)$ exists in (a,b) then there exists $c \in (a,b)$ such that $LRD_n f(c) = 0$.

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I must verify the accuracy of the following information as the article's author.

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