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On the Nörlund-Rice Integral Formula

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***Abstract:** After introducing the famous Nörlund-Rice integral formula, we apply it to Laguerre polynomials, Melzak's relation, and Stirling numbers of the second kind to obtain nice expressions.*

***Keywords:** Stirling numbers, Nörlund-Rice integral, Melzak's identity, Laguerre polynomials.*

I. INTRODUCTION

We have the Nörlund-Rice integral formula [1-6]:

$$\sum_{k=0}^{n} \binom{n}{k} (-1)^k f(k) = \frac{(-1)^n\, n!}{2\pi\, i} \oint_C \frac{f(z)\, dz}{z\,(z-1)(z-2)\cdots(z-n)}, \quad n \geq 0, \qquad (1)$$

where f is analytic in a domain containing the interval $[0, n]$ and C is a closed, simple, positively oriented curve surrounding $[0, n]$.

In Sec. 2 we provide the applications of (1) to Stirling numbers of the second kind, the Melzak's identity, and Laguerre polynomials.

II. APPLICATIONS OF NÖRLUND-RICE'S FORMULA

a).- Let us consider the analytic function $f(z) = z^m,\ \ m \geq 0$ and apply it to (1).

Then (1) gives interesting expressions for the Stirling numbers of the second kind [7-9][20][21][22]:

$$S_m^{[n]} = \frac{(-1)^n}{n!} \sum_{k=0}^{n} \binom{n}{k} (-1)^k k^m = \frac{1}{2\pi\, i} \oint_C \frac{z^m\ dz}{z\,(z-1)\cdots(z-n)}, \quad m, n \geq 0, \qquad (2)$$

thus:

$$S_0^{[0]} = \frac{1}{2\pi\, i} \oint_C \frac{dz}{z} = 1, \quad S_m^{[0]} = \frac{1}{2\pi\, i} \oint_C z^{m-1}\ dz = 0,\ \ m \geq 1, S_m^{[1]} = \frac{1}{2\pi\, i} \oint_C \frac{z^{m-1}}{z-1}\ dz = \begin{cases} 0, & m = 0, \\ 1, & m \geq 1, \end{cases}$$

A. Theorem

The Stirling's numbers of second kind satisfies the recurrence relation $S_n^{[k-1]} = S_{n+1}^{[k]} - k\, S_n^{[k]}$ (3)

Proof: Using the expression (2) we obtain the following (see **[8]**)

$$S_n^{[k-1]} = \frac{1}{2\pi\, i} \oint_C \frac{z^{n-1}}{(z-1)\cdots(z-(k-1))} \cdot= \frac{1}{2\pi\, i} \oint_C \frac{z^n\ dz}{(z-1)\cdots(z-k)} - k\, \frac{1}{2\pi\, i} \oint_C \frac{z^{n-1}\ dz}{(z-1)\cdots(z-k)},$$
$$= S_{n+1}^{[k]} - k\, S_n^{[k]}, \quad q.e.d.$$

There is a well known generating function for the Stirling numbers of the second kind [8]. Upon using that generating function, (2) implies the property:

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$$\sum_{m=0}^{\infty} S_m^{[n]} \frac{q^m}{m!} = \frac{1}{n!}(e^q-1)^n = \frac{1}{2\pi i}\oint_C \frac{e^{zq}\;dz}{z\,(z-1)\cdots(z-n)}\,, \qquad (4)$$

Further from (2) we can Obtain the Following Compact Expression:

$$S_m^{[n]} = \frac{(-1)^n}{n!}(1+\phi)^n, \qquad \phi^k \coloneqq (-1)^k k^m\,. \qquad (5)$$

b).- Let us now consider the function $f(z)=\frac{x^z}{z!}$. Applying this function in (1) we obtain the Laguerre polynomials [10-15] as given below:

$$L_n(x) = \sum_{k=0}^{n}\binom{n}{k}(-1)^k\frac{x^k}{k!} = \frac{(-1)^n\,n!}{2\pi i}\oint_C \frac{\frac{x^z}{z!}\;dz}{z\,(z-1)\cdots(z-n)} = (1+\beta)^n\,, \qquad \beta^k \coloneqq \frac{(-x)^k}{k!}, \qquad (6)$$

From (6), we see that

$$L_0(x)=1, \quad L_1(x)=1-x, \quad L_2(x)=\frac{1}{2!}(2-4x+x^2), \quad L_3(x)=\frac{1}{3!}(6-18x+9x^2-x^3), \ldots;$$

Also from (6), we can deduce the corresponding generating function:

$$\sum_{n=0}^{\infty} L_n(x)\,q^n = \sum_{n=0}^{\infty}[\,(1+\beta)\,q\,]^n = \frac{1}{1-(1+\beta)\,q} = \frac{1}{1-q}\cdot\frac{1}{1-\frac{q\,\beta}{1-q}} = \frac{1}{1-q}\left[1+\frac{q\beta}{1-q}+\left(\frac{q\beta}{1-q}\right)^2+\cdots\right], \qquad (7)$$

$$= \frac{1}{1-q}\left[1+\frac{q}{1-q}\left(-\frac{x}{1!}\right)+\left(\frac{q}{1-q}\right)^2\left(\frac{x^2}{2!}\right)+\left(\frac{q}{1-q}\right)^3\left(-\frac{x^3}{3!}\right)+\cdots\right] = \frac{1}{1-q}\,e^{-\frac{q\,x}{1-q}}\,,$$

c).- Melzak's identity [6, 8, 16-19]:

We now consider the function $f(x+y) = x\binom{x+n}{n}\sum_{k=0}^{n}\binom{n}{k}(-1)^k\frac{f(y-k)}{x+k}$, $x,y\in\mathbb{C}$, $n\geq 0$, $x\neq 0,-1,-2,\ldots,-n$, (8) or any algebraic polynomial $f(t)$ up to degree *n*;

Then from (1) and (8) we obtain

$$f(x+y) = x\binom{x+n}{n}\frac{(-1)^n\,n!}{2\pi i}\oint_C \frac{\frac{f(y-z)}{x+z}}{z\,(z-1)\cdots(z-n)}\,dz\,, \qquad x\neq 0,-1,\ldots,-n, \qquad (9)$$

for example, from (9) with $f(t)=1$ we obtain

$$1 = x\binom{x+n}{n}\frac{(-1)^n\,n!}{2\pi i}\oint_C \frac{\frac{1}{x+z}}{z\,(z-1)\cdots(z-n)}\,dz \overset{(2)}{=} \binom{x+n}{n}(-1)^n\,n!\sum_{j=n}^{\infty}\frac{(-1)^j}{x^j}S_j^{[n]},$$

which simplifies to the Melzak's identity given by

$$\sum_{k=n}^{\infty}\frac{(-1)^j}{x^j}S_j^{[n]} = \frac{(-1)^n}{(x+1)(x+2)\cdots(x+n)}\,. \qquad (10)$$

III. CONCLUSION

Using the Nörlund-Rice integral formula as in (1), we have considered some specific functions in three possible cases. In particular, by considering, $f(z)=z^m$, we proved a recurrence relation satisfied by Stirling's numbers of second kind as in theorem 2.1. Using the function $f(z)=\frac{x^z}{z!}$ we have obtained nice expressions related to Laguerre polynomials and we have also used this formula to determine the generating function for such polynomials. Finally by considering $f(x+y) = x\binom{x+n}{n}\sum_{k=0}^{n}\binom{n}{k}(-1)^k\frac{f(y-k)}{x+k}$ we have obtained the Melzak's Identity as described in (10). These results and connections will provide new insights and rich applications of the famous Norlund-Rice Integral formula. By considering more functions like what we have done in this paper, researchers can try to obtain more entertaining results in future.

DECLARATION STATEMENT

After aggregating input from all authors, I must verify the accuracy of the following information as the article's author.

- **Conflicts of Interest/ Competing Interests:** Based on my understanding, this article has no conflicts of interest.

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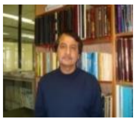

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