OPEN ACCESS

Indian Journal of Advanced Mathematics (IJAM)
ISSN: 2582-8932 (Online), Volume-4 Issue-1, April 2024

Solution of Brocard's Problem

M. I. Karimullah

Abstract: ***Brocard's problem is the solution of the equation, $n! + 1 = m^2$, where m and n are natural numbers. So far only 3 solutions have been found, namely (n,m) = (4,5), (5,11), and (7,71). The purpose of this paper is to show that there are no other solutions. Firstly, it will be shown that if (n,m) is to be a solution to Brocard's problem, then n! = 4AB, where A is even, B is odd, and |A – B| = 1. If n is even (n = 2x) and > 4, it will be shown that necessarily $A = \frac{(2x)!!}{4y}$ and $B = y(2x-1)!!$, for some odd y > 1. Next, it will be shown that x < 2y, and this leads to an inequality in x [namely, $(x(2x-1)!! \pm 1)^2 - 1 - (2x)! < 0$], for which there is no solution when x ≥ 3. If n is odd, there is a similar procedure.***

Keywords: Brocard's Problem, Diophantine Equation, Brown Numbers

I. INTRODUCTION

Brocard's problem is the solution of the Diophantine Equation, $n! + 1 = m^2$, where m and n are natural numbers [5][6][7][8][9]. The problem was posed by Henri Brocard in a pair of articles in 1876 [1] and 1885 [2], and also, independently in 1913 by Srinivasa Ramanujan [3]. As of October 2022, only 3 solutions (aka Brown numbers) have been found, namely (n,m) = (4,5), (5,11), and (7,71); Wikipedia [4]. The purpose of this paper is to show that there are no other solutions.

II. NOTATION USED

The following notations are used.

- N — the set of natural numbers
- n! – the factorial of n
- ∈ — is an element of
- ∀ — for all
- y | x — y divides x
- y ∤ x — y does not divide x
- := — be defined as
- RHS — right-hand side
- LHS — left-hand side
- >> — is much greater than

III. PRELIMINARIES

Assume that $n! + 1 = m^2$.
Note that, n > 1 ⇒ n! is even ⇒ n! + 1 is odd ⇒ m^2 is odd ⇒ m is odd. Hence, m = 2z + 1, where z ∈ N.
That is: $n! + 1 = (2z + 1)^2 \Leftrightarrow n! = (2z + 1)^2 - 1 = (2z)(2z + 2) = (2^2)(z)(z + 1)$. Note that one of the factors, either z or (z + 1), is even while the other is odd.

Hence, if (and only if) the factors in the prime factorization of n! can be partitioned into 3 sets, where one set consists of only 2^2, another set (say A) has an even product, while the third set (say B) has an odd product, and these two products (A and B) differ by 1, then n is a solution of Brocard's problem. Note: A, B, and y, depending on context, will refer to either the **set** of factors or to the **product** of the factors in the set. That is: n is a solution to Brocard's problem if and only if A and B exist, such that n! = 4AB, where A ∈ N, A is even, B ∈ N, B is odd, and |A – B| = 1.

IV. N IS EVEN

Let n be even, say n = 2x and let n > 4.
One possible partition of the factors of n! is
$n! = (2x)! = (2x)!!\,(2x-1)!! = 4\left[\frac{(2x)!!}{4}\right](2x-1)!!$
$\Rightarrow A = \frac{(2x)!!}{4}$ and $B = (2x-1)!!$

4.1:
It will now be shown that such a partition does not result in |A – B| = 1 (when n > 4).

With $A = \frac{(2x)!!}{4}$ and $B = (2x-1)!!$, the table below shows the values of A – B for a set of the first consecutive even (positive) integers.

Values of A – B for First Consecutive Even Integers			
n	**A**	**B**	**A - B**
2	1/2	1	-1/2
4	2	3	-1
6	12	15	-3
8	96	105	-9
10	960	945	15
12	11520	10395	1125

(As shown in the table, |A – B| = 1, when n = 4; meaning that n = 4 solves Brocard's problem.)
It will now be shown that ∀ even $n \geq 10,\ |A - B| \neq 1$, by showing that $A - B > 1$.
This will be proved by mathematical induction, using n = 10 as the base case, where $A - B = 15 > 1$.
Let $g(2r) := A - B = \frac{(2r)!!}{4} - (2r-1)!!$
For the inductive step, assume that when n = 2r (where 2r ≥ 10), g(2r) > 1.
Consider $g(2\{r+1\})$
$= \frac{(2r+2)!!}{4} - (2r+1)!!$
$= \frac{(2r+2)(2r)!!}{4} - (2r+1)(2r-1)!!$
$> \frac{(2r+1)(2r)!!}{4} - (2r+1)(2r-1)!!$
$= (2r+1)\left(\frac{(2r)!!}{4} - (2r-1)!!\right)$
$= (2r+1)g(2r)$
> 1; since (2r + 1) > 1 and, by assumption, g(2r) > 1.

Manuscript received on 09 March 2024 | Revised Manuscript received on 16 March 2024 | Manuscript Accepted on 15 April 2024 | Manuscript published on 30 April 2024.
** Correspondence Author (s)*
Mohamed Imteaz Karimullah*, Burlington, Ontario, Canada. E-mail: moh_imt_kar@hotmail.com. ORCID ID: 0009-0008-7847-3429

4.2:
As shown above, g(2r + 2) > (2r + 1)g(2r) ⇒ as r gets larger, so does $\left[\frac{(2r)!!}{4} - (2r-1)!!\right]$.
Since $|A - B| \gg 1, \forall\, n \geq 10$, it means that if |A – B| is to have a chance of being equal to 1, the factors of n! must be partitioned in the following way: $n! = 4\left[\frac{(2x)!!}{4y}\right][y(2x-1)!!]$, where y (y > 1) is some odd factor(s) in (2x)!!. That is, some odd factor(s) (whose product is y) of A must be transferred to B. Note that y depends on n; i.e. y is a function of x, y = y(x).
Hence, if n is even, n > 4, and n is a solution to Brocard's problem, then
$n! = (2x)! = 4\left[\frac{(2x)!!}{4y}\right][(2x-1)!!\,y]$;
where y is an odd integer, $y > 1$, $\frac{(2x)!!}{4y} \in$ N and $\left|\frac{(2x)!!}{4y} - (2x-1)!!\,y\right| = 1$.

4.3:
It will now be shown that $y > \frac{x}{2}$, if $\left|\frac{(2x)!!}{4y} - (2x-1)!!\,y\right| = 1$
Assume that $y^2 \mid (2x)!!$
$\Rightarrow y \mid \frac{(2x)!!}{4y}$
$\Rightarrow y \mid \left(\frac{(2x)!!}{4y} - y(2x-1)!!\right)$
$\Rightarrow y \mid \pm 1$; an impossibility, since y > 1.
Hence, $y^2 \nmid (2x)!!$
$\Rightarrow y^2 \nmid (2^x.x!)$, since $(2x)!! = 2^x.x!$
$\Rightarrow y^2 \nmid x!$, since y is odd
$\Rightarrow x < 2y$

4.4:
It will now be shown that $(x(2x-1)!! \pm 1)^2 - 1 - (2x)! < 0$, if n = 2x is to be a solution to Brocard's problem.

Consider $\frac{(2x)!!}{4y} - (2x-1)!!\,y = \pm 1$
$\Rightarrow 4(2x-1)!!\,[y(x)]^2 \pm 4y(x) - (2x)!! = 0$; remember y = y(x)
$\Rightarrow [y(x)]^2 \pm \frac{4y(x)}{4(2x-1)!!} - \frac{(2x)!!}{4(2x-1)!!} = 0$; note that $4(2x-1)!! \neq 0$
$\Rightarrow [y(x)]^2 \pm \frac{y(x)}{(2x-1)!!} = \frac{(2x)!!}{4(2x-1)!!}$
$\Rightarrow [y(x)]^2 \pm \frac{y(x)}{(2x-1)!!} + \left(\frac{1}{2(2x-1)!!}\right)^2 = \frac{(2x)!!}{4(2x-1)!!} + \left(\frac{1}{2(2x-1)!!}\right)^2$
$\Rightarrow \left(y(x) \pm \frac{1}{2(2x-1)!!}\right)^2 = \frac{(2x)!!(2x-1)!! + 1}{4[(2x-1)!!]^2}$
$\Rightarrow \left(y(x) \pm \frac{1}{2(2x-1)!!}\right)^2 = \frac{(2x)! + 1}{4[(2x-1)!!]^2}$
$\Rightarrow y(x) \pm \frac{1}{2(2x-1)!!} = +\frac{\sqrt{(2x)!+1}}{2(2x-1)!!}, -\frac{\sqrt{(2x)!+1}}{2(2x-1)!!}$
$\Rightarrow y(x) = \mp\frac{1}{2(2x-1)!!} + \frac{\sqrt{(2x)!+1}}{2(2x-1)!!}, \mp\frac{1}{2(2x-1)!!} - \frac{\sqrt{(2x)!+1}}{2(2x-1)!!}$
$\Rightarrow y(x) = \frac{\mp 1 + \sqrt{(2x)!+1}}{2(2x-1)!!}, \frac{\mp 1 - \sqrt{(2x)!+1}}{2(2x-1)!!}$
$\Rightarrow y = \frac{\mp 1 + \sqrt{1+(2x)!}}{2(2x-1)!!}$; since y > 0 and $\left(\mp 1 - \sqrt{(2x)! + 1}\right) < 0$
$\Rightarrow x < 2\left(\frac{\mp 1 + \sqrt{1+(2x)!}}{2(2x-1)!!}\right)$; since x < 2y
$\Rightarrow x(2x-1)!! < \mp 1 + \sqrt{1+(2x)!}$; note that $(2x-1)!! > 0$
$\Rightarrow x(2x-1)!! \pm 1 < \sqrt{1+(2x)!}$
[Note that n = 2x > 4 ⇒ x > 2 ⇒ the LHS of the above inequality is > 0.]
$\Rightarrow (x(2x-1)!! \pm 1)^2 < 1 + (2x)!$
$\Rightarrow (x(2x-1)!! \pm 1)^2 - 1 - (2x)! < 0$

4.5:
It will now be shown that $(x(2x-1)!! \pm 1)^2 - 1 - (2x)! < 0 \Rightarrow x \ngeq 3$.
Let $g(2x) := (x(2x-1)!! + 1)^2 - 1 - (2x)!$
and let $h(2x) := (x(2x-1)!! - 1)^2 - 1 - (2x)! = x(2x-1)!!\,\{x(2x-1)!! - 2\} - (2x)!$
Hence, a necessary but not sufficient condition for 2x to solve Brocard's problem is $g(2x) < 0$ or $h(2x) < 0$.
It will now be shown that $h(2x) > 0, \forall\, 2x \geq 6$.
This will be proved by mathematical induction, using 2x = 6 as the base case, where h(6) = 1215 > 0.
For the inductive step, assume that when n = 2r (where 2r ≥ 6), h(2r) > 0.
Consider $h(2\{r+1\})$
$= (r+1)(2r+1)!!\,\{(r+1)(2r+1)!! - 2\} - (2r+2)!$
$= (r+1)(2r+1)(2r-1)!!\,\{(r+1)(2r+1)!! - 2\} - (2r+2)(2r+1)(2r)!$
$= (2r+1)[(r+1)(2r-1)!!\,\{(r+1)(2r+1)!! - 2\} - (2r+2)(2r)!]$
$= (2r+1)[(r+1)(2r-1)!!\,\{(r+1)(2r+1)(2r-1)!! - 2\} - (2r+2)(2r)!]$
$> (2r+1)[(r+1)(2r-1)!!\,\{r(2r+2)(2r-1)!! - 2\} - (2r+2)(2r)!]$
[since $(r+1)(2r+1) > r(2r+2)$, when $2r \geq 6$.]
$> (2r+1)[(r+1)(2r-1)!!\,\{r(2r+2)(2r-1)!! - 2(2r+2)\} - (2r+2)(2r)!]$
[since $-2 > -2(2r+2)$, when $2r \geq 6$; and also $(2r+1)(r+1)(2r-1)!! > 0$.]
$= (2r+1)(2r+2)[(r+1)(2r-1)!!\,\{r(2r-1)!! - 2\} - (2r)!]$
$> (2r+1)(2r+2)[r(2r-1)!!\,\{r(2r-1)!! - 2\} - (2r)!]$
[since $r+1 > r$, when $2r \geq 6$.]
$= (2r+1)(2r+2)h(2r)$
> 0; since (2r + 1)(2r + 2) > 0 and, by assumption, h(2r) > 0.
Note that g(2x)
$= (x(2x-1)!! + 1)^2 - 1 - (2x)!$
$> (x(2x-1)!! - 1)^2 - 1 - (2x)!$
$= h(2x)$
Hence, $g(2x) > 0, \forall\, 2x \geq 6$.

V. N IS ODD

Let n be odd, say n = 2x + 1 and let n ≥ 5.
One possible partition of the factors of n! is
$n! = (2x+1)! = (2x+1)!!\,(2x)!! = 4\left[\frac{(2x)!!}{4}\right](2x+1)!!$
$\Rightarrow A = \frac{(2x)!!}{4}$ and $B = (2x+1)!!$

5.1:
It will now be shown that such a partition does not result in |A – B| = 1 (when n ≥ 5).
With $A = \frac{(2x)!!}{4}$ and $B = (2x+1)!!$, the table below shows the values of A – B for a set of the first consecutive odd (positive) integers.

Values of A – B for First Consecutive Odd Integers			
n	A	B	A - B
1	0.25	1	-0.75
3	0.5	3	-2.5
5	2	15	-13
7	12	105	-93
9	96	945	-849

It will now be shown that $\forall\, n \geq 5,\ |A - B| \neq 1$, by showing that $A - B < -1$. This will be proved by mathematical induction; using n = 5 as the base case, where $A - B = -13 < -1$.

Let $g(2r+1) := A - B = \frac{(2r)!!}{4} - (2r+1)!!$

For the inductive step, assume that when n = 2r + 1 (where 2r + 1 ≥ 5), g(2r + 1) < -1.

Consider $g(2\{r+1\}+1)$

$= \frac{(2r+2)!!}{4} - (2r+3)!!$

$= \frac{(2r+2)(2r)!!}{4} - (2r+3)(2r+1)!!$

$< \frac{(2r+3)(2r)!!}{4} - (2r+3)(2r+1)!!$

$= (2r+3)\left(\frac{(2r)!!}{4} - (2r+1)!!\right)$

$= (2r+3)g(2r+1)$

< -1; since (2r + 3) > 1 and, by assumption, g(2r + 1) < -1.

5.2:

As shown above, g(2r + 3) < (2r + 3)g(2r+1) with g(2r + 1) < -1 ⇒ as r gets larger, so does $\left|\frac{(2r)!!}{4} - (2r+1)!!\right|$.

Since $|A - B| \gg 1, \forall\, n \geq 5$, it means that if |A – B| is to have a chance of being equal to 1, the factors of n! must be partitioned in the following way: $n! = 4\left[\frac{(2x)!!y}{4}\right]\left[\frac{(2x+1)!!}{y}\right]$,

where y (y > 1) is some odd factor(s) in (2x+1)!!. That is, some odd factor(s) (whose product is y) of B must be transferred to A. Note that y depends on n; i.e. y is a function of x, y = y(x).

Hence, if n is odd, n ≥ 5, and n is a solution to Brocard's problem, then

$n! = (2x+1)! = 4\left[\frac{(2x)!!}{4}y\right]\left[\frac{(2x+1)!!}{y}\right]$;

where y is an odd integer, $y > 1$, $\frac{(2x+1)!!}{y} \in$ N and

$\left|\frac{(2x)!!}{4}y - \frac{(2x+1)!!}{y}\right| = 1$.

5.3:

It will now be shown that $y > \frac{2x+1}{3}$, if $\left|\frac{(2x)!!}{4}y - \frac{(2x+1)!!}{y}\right| = 1$.

Assume that $y^2 \mid (2x+1)!!$

$\Rightarrow y \mid \frac{(2x+1)!!}{y}$

$\Rightarrow y \mid \left(\frac{(2x)!!}{4}y - \frac{(2x+1)!!}{y}\right)$

$\Rightarrow y \mid \pm 1$; an impossibility, since y > 1.

Hence, $y^2 \nmid (2x+1)!!$

Note that, (2x + 1)!!, when expanded, gives only odd factors.

If (2x + 1) ≥ 3y, then y^2 | (2x + 1)!!

Thus, 2x + 1 < 3y.

5.4:

It will now be shown that $\left(\frac{(2x+1)(2x)!!}{6} \mp 1\right)^2 - 1 - (2x+1)! < 0$ if n = 2x + 1 is to be a solution to Brocard's problem.

Consider $\frac{(2x)!!}{4}y(x) - \frac{(2x+1)!!}{y(x)} = \pm 1$

$\Rightarrow (2x)!!\,[y(x)]^2 \mp 4y(x) - 4(2x+1)!! = 0$

$\Rightarrow [y(x)]^2 \mp \frac{4y(x)}{(2x)!!} - \frac{4(2x+1)!!}{(2x)!!} = 0$; Note that $(2x)!! \neq 0$

$\Rightarrow [y(x)]^2 \mp \frac{4y(x)}{(2x)!!} = \frac{4(2x+1)!!}{(2x)!!}$

$\Rightarrow [y(x)]^2 \mp \frac{4y(x)}{(2x)!!} + \left(\frac{2}{(2x)!!}\right)^2 = \frac{4(2x+1)!!}{(2x)!!} + \left(\frac{2}{(2x)!!}\right)^2$

$\Rightarrow \left(y(x) \mp \frac{2}{(2x)!!}\right)^2 = \frac{4(2x+1)!!(2x)!!+4}{[(2x)!!]^2}$

$\Rightarrow \left(y(x) \mp \frac{2}{(2x)!!}\right)^2 = \frac{4[(2x+1)!+1]}{[(2x)!!]^2}$

$\Rightarrow y(x) \mp \frac{2}{(2x)!!} = +\frac{2\sqrt{(2x+1)!+1}}{(2x)!!}, -\frac{2\sqrt{(2x+1)!+1}}{(2x)!!}$

$\Rightarrow y(x) = \pm\frac{2}{(2x)!!} + \frac{2\sqrt{(2x+1)!+1}}{(2x)!!}, \pm\frac{2}{(2x)!!} - \frac{2\sqrt{(2x+1)!+1}}{(2x)!!}$

$\Rightarrow y(x) = \frac{\pm2+2\sqrt{(2x+1)!+1}}{(2x)!!}, \frac{\pm2-2\sqrt{(2x+1)!+1}}{(2x)!!}$

$\Rightarrow y(x) = \frac{\pm2+2\sqrt{(2x+1)!+1}}{(2x)!!}$; since y > 0 and $\left(\pm2 - 2\sqrt{(2x+1)!+1}\right) < 0$

$\Rightarrow \frac{2x+1}{3} < \frac{\pm2+2\sqrt{(2x+1)!+1}}{(2x)!!}$; since $y > \frac{2x+1}{3}$

$\Rightarrow \frac{(2x+1)(2x)!!}{6} < \pm1 + \sqrt{(2x+1)!+1}$; note that $(2x)!! > 0$

$\Rightarrow \frac{(2x+1)(2x)!!}{6} \mp 1 < \sqrt{1+(2x+1)!}$

[Note that n = 2x + 1 ≥ 5 ⇒ x ≥ 2 ⇒ the LHS of the above inequality is > 0.]

$\Rightarrow \left(\frac{(2x+1)(2x)!!}{6} \mp 1\right)^2 < 1 + (2x+1)!$

$\Rightarrow \left(\frac{(2x+1)(2x)!!}{6} \mp 1\right)^2 - 1 - (2x+1)! < 0$

5.5:

It will now be shown that $\left(\frac{(2x+1)(2x)!!}{6} \mp 1\right)^2 - 1 - (2x+1)! < 0 \Rightarrow x \ngeq 5$

Let $g(2x+1) := \left(\frac{(2x+1)(2x)!!}{6} + 1\right)^2 - 1 - (2x+1)!$

and let $h(2x+1) := \left(\frac{(2x+1)(2x)!!}{6} - 1\right)^2 - 1 - (2x+1)!$

$= \frac{(2x+1)(2x)!!}{6}\left(\frac{(2x+1)(2x)!!}{6} - 2\right) - (2x+1)!$

Hence, a necessary but not sufficient condition for 2x + 1 to solve Brocard's problem is $g(2x+1) < 0$ or $h(2x+1) < 0$.

It will now be shown that $h(2x+1) > 0, \forall\, (2x+1) \geq 11$. This will be proved by mathematical induction, using 2x + 1 = 11 as the base case, where h(11) = 9630720 > 0.

For the inductive step, assume that when n = 2r + 1 (where 2r + 1 ≥ 11), h(2r + 1) > 0.

Consider $h(2\{r+1\}+1)$

$= \frac{(2\{r+1\}+1)(2\{r+1\})!!}{6}\left(\frac{(2\{r+1\}+1)(2\{r+1\})!!}{6} - 2\right) - (2\{r+1\}+1)!$

$= \frac{(2r+3)(2r+2)!!}{6}\left(\frac{(2r+3)(2r+2)!!}{6} - 2\right) - (2r+3)!$

$= \frac{(2r+3)(2r+2)(2r)!!}{6}\left(\frac{(2r+3)(2r+2)!!}{6} - 2\right) - (2r+3)(2r+2)(2r+1)!$

$= (2r+3)(2r+2)\left[\frac{(2r)!!}{6}\left(\frac{(2r+3)(2r+2)!!}{6} - 2\right) - (2r+1)!\right]$

$> (2r+3)(2r+2)\left[\frac{(2r)!!}{6}\left(\frac{(2r+3)(2r+2)!!}{6} - 2(2r+2)\right) - (2r+1)!\right]$

Retrieval Number:100.1/ijam.B117404021024
DOI: 10.54105/ijam.B1174.04010424
Journal Website: www.ijam.latticescipub.com

Published By:
Lattice Science Publication (LSP)

[since $-2 > -2(2r+2)$, when $2r+1 \geq 11$; and also $(2r+3)(2r+2)\frac{(2r)!!}{6} > 0$.]

$= (2r+3)(2r+2)\left[\frac{(2r)!!}{6}\left(\frac{(2r+3)(2r+2)(2r)!!}{6} - 2(2r+2)\right) - (2r+1)!\right]$

$= (2r+3)(2r+2)\left[\frac{(2r+2)(2r)!!}{6}\left(\frac{(2r+3)(2r)!!}{6} - 2\right) - (2r+1)!\right]$

$> (2r+3)(2r+2)\left[\frac{(2r+1)(2r)!!}{6}\left(\frac{(2r+3)(2r)!!}{6} - 2\right) - (2r+1)!\right]$

[since $2r + 2 > 2r + 1$, when$2r+1 \geq 11$.]

$> (2r+3)(2r+2)\left[\frac{(2r+1)(2r)!!}{6}\left(\frac{(2r+1)(2r)!!}{6} - 2\right) - (2r+1)!\right]$

[since $2r + 3 > 2r + 1$, when$2r+1 \geq 11$.]

$= (2r+3)(2r+2)h(2r+1)$

> 0; since $(2r + 3)(2r + 2) > 0$ and, by assumption, $h(2r + 1) > 0$.

Note that $g(2x+1)$

$= \left(\frac{(2x+1)(2x)!!}{6} + 1\right)^2 - 1 - (2x+1)!$

$> \left(\frac{(2x+1)(2x)!!}{6} - 1\right)^2 - 1 - (2x+1)!$

$= h(2x+1)$

Hence, $g(2x+1) > 0, \forall\, 2x+1 \geq 11$.

VI. CONCLUSION

6.1:

If n is even, $n > 4$, and n is a solution to Brocard's problem, then

$n! = (2x)! = 4\left[\frac{(2x)!!}{4y}\right][(2x-1)!!\,y]$;

where y is an odd integer, $y > 1$, $\frac{(2x)!!}{4y} \in N$ and $\left|\frac{(2x)!!}{4y} - (2x-1)!!\,y\right| = 1$.

The above implies $y > x/2$; which, in turn, implies $(x(2x-1)!! \pm 1)^2 - 1 - (2x)! < 0$; which, in turn, implies $x \ngeq 3$.

Therefore, there is no even integer ≥ 6, which satisfies the above necessary condition to solve Brocard's problem.

6.2:

If n is odd, $n \geq 5$, and n is a solution to Brocard's problem, then

$n! = (2x+1)! = 4\left[\frac{(2x)!!}{4}y\right]\left[\frac{(2x+1)!!}{y}\right]$;

where y is an odd integer, $y > 1$, $\frac{(2x+1)!!}{y} \in N$ and $\left|\frac{(2x)!!}{4}y - \frac{(2x+1)!!}{y}\right| = 1$

The above implies $y > \frac{2x+1}{3}$; which, in turn, implies $\left(\frac{(2x+1)(2x)!!}{6} \mp 1\right)^2 - 1 - (2x+1)! < 0$; which, in turn, implies $x \ngeq 5$.

Therefore, there is no odd integer ≥ 11, which satisfies the above necessary condition to solve Brocard's problem. As an aside: to get the solutions corresponding to n = 5 and n = 7, y = 3 in each case.

DECLARATION STATEMENT

Funding	No, I did not receive any funding.
Conflicts of Interest	No conflicts of interest.
Ethical Approval and Consent to Participate	No, the article does not require ethical approval and consent to participate with evidence.
Availability of Data and Material	Not relevant.
Authors Contributions	I am only the sole author of the article.

REFERENCES

1. Brocard, H. (1876), "Question 166", Nouv. Corres. Math., 2: 287
2. Brocard, H. (1885), "Question 1532", Nouv. Ann. Math., 4: 391
3. Ramanujan, S. (2000), "Question 469", in Hardy, G. H.; Aiyar, P. V. Seshu; Wilson, B. M. (eds.), Collected papers of Srinivasa Ramanujan, Providence, Rhode Island: AMS Chelsea Publishing, p. 327, ISBN 0-8218-2076-1, MR 2280843
4. Wikipedia: https://en.wikipedia.org/wiki/Brocard%27s_problem
5. Prof. J. López-Bonilla, & R. Sivaraman. (2024). On Solving a Quadratic Diophantine Equation Involving Odd Powers of 17. In Indian Journal of Advanced Mathematics (Vol. 4, Issue 1, pp. 1–3). https://doi.org/10.54105/ijam.a1165.04010424
6. Moonchaisook, V. (2021). On the Solutions of Diophantine Equation (Mp − 2) x + (Mp + 2) y = z 2 where Mp is Mersenne Prime. In International Journal of Basic Sciences and Applied Computing (Vol. 3, Issue 4, pp. 1–3). https://doi.org/10.35940/ijbsac.d0216.083421
7. Tahiliani, Dr. S. (2021). More on Diophantine Equations. In International Journal of Management and Humanities (Vol. 5, Issue 6, pp. 26–27). https://doi.org/10.35940/ijmh.l1081.025621
8. Yegnanarayanan, V., Narayanan, V., & Srikanth, R. (2019). On Infinite Number of Solutions for one type of Non-Linear Diophantine Equations. In International Journal of Innovative Technology and Exploring Engineering (Vol. 9, Issue 1, pp. 1665–1669). https://doi.org/10.35940/ijitee.a4706.119119
9. Solutions to Non-Linear Diophantine Equation Pχ+ (P+5)y =z2 with P is Mersenne Prime. (2019). In International Journal of Recent Technology and Engineering (Vol. 8, Issue 2S7, pp. 237–238). https://doi.org/10.35940/ijrte.b1060.0782s719

AUTHOR PROFILE

M. I. Karimullah MSc, BSc. He has a BSc in Physics from the University of London and an MSc in Mathematics from the University of Toronto. He was a 1980 Guyana Scholar and the recipient of the following academic awards: Government of Guyana Scholarship, European Commission Scholarship, and University of Toronto Research Assistantship. He was a lecturer in Mathematics and Physics at the University of Guyana, and in Mathematics and Computer Science at the University of The Bahamas. Also, he was also a Teaching Assistant at the University of Toronto and an Examiner in Physics at the Caribbean Examinations Council. His certificates in Information Technology include VMware Certified Advanced Professional, EMC Proven Associate, RedHat Certified Engineer, Cisco Certified Network Associate, Certified Novell Engineer, and Microsoft Certified Systems Engineer, Microsoft Certified Database Administrator, and Microsoft Certified Solution.