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On Some Relations Involving the Ramanujan's Tau Function

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***Abstract:** It is known a recurrence relation for the Ramanujan's tau-function involving the sum of divisors function$\sigma(n)$, whose solution gives a closed formula for $\tau(n)$ in terms ofcomplete Bell polynomials, and a determinantal expression for $\sigma(m)$ where participate the values $\tau(k)$.*

***Keywords:** Sum of divisors function, Recurrence relations, Ramanujan's function $\tau(n)$, Bell polynomials, Chebyshev polynomials of the second kind.*

I. INTRODUCTION

We know the following recurrence relation for the Ramanujan's tau-function [1, 2]:

$$n\,\tau(n+1) = -24\,\sum_{j=1}^{n} \sigma(j)\,\tau(n+1-j), \quad n \geq 1, \qquad (1)$$

which allows an easy recursive manner to calculate the values of $\tau(m)$: 1, -24, 252, -1472, 4830, - 6048,…, that is, the sequence A000594 [3]. Besides, this function verifies interesting properties if p is a prime number [1, 2, 4-8]:

$$\tau(p^{n+2}) = \tau(p)\,\tau(p^{n+1}) - p^{11}\,\tau(p^n), \qquad n \geq 0, \qquad (2)$$

$$|\tau(m)| \leq m^{\frac{11}{2}}\,d(m) \therefore |\tau(p)| \leq 2\,p^{\frac{11}{2}}, \qquad (3)$$

where $d(m)$is the number of divisors of m.

In Sec. 2 we show that (1) gives two options: To write $\tau(n)$ in terms of $\sigma(m)$ via the complete Bell polynomials [9-15][21][22], or to express $\sigma(n)$ as a determinant whose entries are values of the tau function. In Sec. 3we use (2), (3) and the Chebyshev polynomials [16] to obtain a formula of Ramanujan [1][18] for$\tau(p^n)$.

II. EXPLICIT SOLUTIONS OF (1)

From (1) it is immediate a closed expression for the Ramanujan's tau-function in terms of the complete Bell polynomials [15]:

$$\tau(n+1) = \frac{1}{n!} B_n\big(-24\cdot 0!\,\sigma(1), -24\cdot 1!\,\sigma(2), -24\cdot 2!\,\sigma(3), \ldots, -24\cdot(n-1)!\,\sigma(n)\big), \qquad n \geq 0, \qquad (4)$$

which also allows reproduce the sequence of integers A000594., or equivalently:

$$\tau(n+1) = \sum_{k=0}^{n} \frac{(-24)^k}{k!} C_{n-k}^{(k)}, \qquad C_r^{(0)} = \delta_{0r}, \qquad C_r^{(1)} = \frac{\sigma(r+1)}{r+1}, \qquad C_0^{(r)} = 1, \qquad (5)$$

$$j\,C_j^{(r)} = \sum_{m=1}^{j} \frac{[m\,(r+1) - j]\,\sigma(m+1)}{m+1} C_{j-m}^{(r)}.$$

From (1) we can to employ determinants to obtain the sum of divisors function in terms of the tau function:

$$\sigma(n) = -\frac{1}{24}\begin{vmatrix} n\tau(n+1) & \tau(2) & \tau(3) & \tau(4) & \cdots & \tau(n) \\ (n-1)\tau(n) & 1 & \tau(2) & \tau(3) & \cdots & \tau(n-1) \\ \vdots & 0 & 1 & \tau(2) & \cdots & \tau(n-2) \\ \vdots & \vdots & \vdots & 1 & \vdots & \vdots \\ 2\tau(3) & 0 & 0 & \vdots & \vdots & \vdots \\ \tau(2) & 0 & 0 & 0 & 0 & 1 \end{vmatrix}, \qquad (6)$$

that is:

$$\sigma(1) = -\frac{1}{24}|\tau(2)|, \quad \sigma(2) = -\frac{1}{24}\begin{vmatrix} 2\tau(3) & \tau(2) \\ \tau(2) & 1 \end{vmatrix}, \quad \sigma(3) = -\frac{1}{24}\begin{vmatrix} 3\tau(4) & \tau(2) & \tau(3) \\ 2\tau(3) & 1 & \tau(2) \\ \tau(2) & 0 & 1 \end{vmatrix}, \ldots \qquad (7)$$

which it is equivalent to:

$$\sigma(n) = \frac{n}{24}\sum_{j=1}^{n} \frac{(-1)^j}{j} A_{n-j}^{(j)}, \qquad A_k^{(0)} = \delta_{0k}, \qquad A_k^{(1)} = \tau(k+2), \qquad A_0^{(k)} = (-24)^k, \qquad (8)$$

$$j\,A_j^{(r)} = -\frac{1}{24}\sum_{m=1}^{j} [m(r+1) - j]\,\tau(m+2)A_{j-m}^{(r)}.$$

III. RAMANUJAN'S FORMULA FOR $\tau(p^n)$

In (3) we can use $m = p^n$, where p is a prime number, thus$|\tau(p^n)| \leq (n+1)p^{\frac{11\,n}{2}}$, then it is natural to work with the expression:

$$\frac{\tau(p^n)}{n+1} = Q_n(p)p^{\frac{11\,n}{2}}, \qquad |Q_n(p)| \leq 1, \qquad (9)$$

hence $Q_1(p) = \frac{\tau(p)}{2\,p^{\frac{11}{2}}}$ verifying the property (3) proved by Deligne [5][19][20]. We can employ (9) in the recurrence relation (2) to obtain:

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$$(n+3)Q_{n+2} = 2\,(n+2)\,Q_1 Q_{n+1} - (n+1)\,Q_n\,, \qquad (10)$$

whose comparison with the recurrence relation satisfied by the Chebyshev polynomials of the second kind [16]:

$$U_{n+2}(\cos\theta) = 2\cos\theta\, U_{n+1}(\cos\theta) - U_n(\cos\theta), \qquad (11)$$

implies the connections:

$$\cos\theta = Q_1 = \frac{\tau(p)}{2\,p^{\frac{11}{2}}}, \qquad U_n(\cos\theta) = (n+1)Q_n(p) = \frac{\sin(n+1)\theta}{\sin\theta}, \qquad (12)$$

verifying the inequality (9) because we know that $|U_n(\cos\theta)| \le (n+1)$. Finally, (9) and (12) generate the following formula published by Ramanujan [1, 2]:

$$\tau(p^n) = \frac{Sin\,(n+1)\theta_p}{Sin\,\theta_p}\, p^{\frac{11\,n}{2}}\,. \qquad (13)$$

Remark: We note that (2) implies the property:

$$\tau(4n) = 3[-8\,\tau(2n) - 683\,\tau(n)] + \tau(n), \qquad (14)$$

therefore $\tau(4n) \equiv \tau(n) \pmod 3$ [17], and:

$$\tau(4n) = 8\,[-3\,\tau(2n) - 256\,\tau(n)] \quad \therefore \quad \tau(4n) \equiv 0 \pmod k,\; k = 2, 4, 8. \qquad (15)$$

IV. CONCLUSION

Though there are several ways of expressing Ramanujan's Tau function using polynomials, special functions and various tools in mathematics, in this paper, we have expressed the sum of divisors function as a determinant whose entries involves Tau function values as in (6). The first three values are explicitly arrived in (7). A more general form of these expressions are provided in (8). Finally using the Tau conjectures proposed by Ramanujan and using Chebyshev polynomials, we have deduced some interesting congruence related to modulo 2, 4 and 8 as provided in (15). These little observations may provide new insight upon knowing the values of Ramanujan Tau function.

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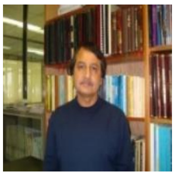

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