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Ramanujan's Tau-Function in Terms of Bell Polynomials

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***Abstract:** We obtain a recurrence relation for the Ramanujan's tau-function involving the sum of divisors function, and the solution of this recurrence gives a closed formula for $\tau(n)$ in terms of the complete Bell polynomials.*

***Keywords:** Sum of divisors function, Color partitions, Recurrence relations, Complete Bell polynomials, Ramanujan's function $\tau(n)$.*

I. INTRODUCTION

We know the following recurrence relation [1-3][29][30]:

$$n\, p_k(n) = -k \sum_{j=1}^{n} \sigma(j)\, p_k(n-j), \qquad n \geq 1, \tag{1}$$

where σ is the sum of divisors function [4-8][31] and $p_k(n)$ is the number of color partitions of n [9-12]. The solution of (1) is given by [3]:

$$p_k(n) = \frac{1}{n!} B_n\big(-k \cdot 0!\, \sigma(1), -k \cdot 1!\, \sigma(2), -k \cdot 2!\, \sigma(3), \ldots, -k \cdot (n-1)!\, \sigma(n)\big), \tag{2}$$

in terms of the complete Bell polynomials [13-19]. In Sec. 2 we use (1) and (2) for the case $k = 24$ to obtain a recurrence relation verified by the Ramanujan's tau-function and its corresponding closed expression via Bell polynomials.

II. RAMANUJAN'S FUNCTION $\tau(n)$ [20]

We have the connection:

$$p_{24}(n) = \tau(n+1), \tag{3}$$

then (1) implies the following recurrence relation for the Ramanujan's tau-function:

$$n\, \tau(n+1) = -24 \sum_{j=1}^{n} \sigma(j)\, \tau(n+1-j), \quad n \geq 1, \tag{4}$$

which gives an easy recursive manner to determine the values of $\tau(m)$:1, -24, 252, -1472, 4830, -6048,…, that is, the sequence A000594 [21].The property (4) is an alternative to the computationally efficient triangular recurrence formula [20, 22-24]:

$$(n-1)\, \tau(n) = \sum_{m=1}^{\lfloor b_n \rfloor} (-1)^{m+1} (2m+1) \left(n - 1 - \frac{9}{2} m(m+1)\right) \tau\left(n - \frac{1}{2} m(m+1)\right), \tag{5}$$

where $b_n = \frac{1}{2}\left(\sqrt{8n+1} - 1\right)$.

From (2) and (3) it is immediate a closed expression for the Ramanujan's tau-function in terms of the complete Bell polynomials:

$$\tau(n+1) = \frac{1}{n!} B_n\big(-24 \cdot 0!\, \sigma(1), -24 \cdot 1!\, \sigma(2), -24 \cdot 2!\, \sigma(3), \ldots, -24 \cdot (n-1)!\, \sigma(n)\big), \qquad n \geq 0, \tag{6}$$

which also allows reproduce the sequence of integers A000594; we can consider to (6) as an alternative to several expressions in the literature, for example [25, 26]:

$$\tau(n) = n^4\, \sigma(n) - 24 \sum_{k=1}^{n-1} k^2 \left(35\, k^2 - 52\, k\, n + 18\, n^2\right) \sigma(k)\, \sigma(n-k), \qquad n \geq 1, \tag{7}$$

or for the closed relations [24, 27]:

$$\tau(n) = 8000\, \big((\sigma_3 * \sigma_3) * \sigma_3\big)(n) - 147\, (\sigma_5 * \sigma_5)(n), \qquad \sigma_r(m) = \sum_{d/m} d^r, \qquad m \geq 1, \tag{8}$$

$$= \frac{65}{756} \sigma_{11}(n) + \frac{691}{756} \sigma_5(n) - \frac{691}{3} \sum_{k=1}^{n-1} \sigma_5(k)\, \sigma_5(n-k), \quad \sigma_3(0) = \frac{1}{240}, \quad \sigma_5(0) = -\frac{1}{504}, \tag{9}$$

where $*$ denotes Cauchy convolution [6].

Remark: The relations (1) and (4) were obtained by Gandhi [9, 28] and Ramanujan [20], respectively.

III. CONCLUSION

The main aim of this paper is to obtain the closed form of Ramanujan's Tau function in terms of complete Bell Polynomials. For this, we considered a recurrence relation connecting colored partitions in terms of sum of divisor function of a positive integer n as in (1). The solution of this recurrence relation in terms of complete Bell polynomials is presented in (2).

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By considering the case for $k = 24$, we connect colored partitions of n with Ramanujan's Tau function as in (3). Using this, we have connected Ramanujan's Tau function with complete Bell polynomials as in (6). Now using expression (6), we have found an expression for Ramanujan's Tau function in terms of divisor function using Cauchy convolution product as presented in (8) and (9). These expressions provide an alternate view of expressing Ramanujan's Tau function in terms of simple arithmetic functions.

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REFERENCES

1. G. E. Andrews, S. Kumar Jha, J. López-Bonilla, *Sums of squares, triangular numbers, and divisor sums,* J. of Integer Sequences **26** (2023) Article 23.2.5
2. M. A. Pathan, H. Kumar, M. Muniru Iddrisu, J. López-Bonilla, *Polynomial expressions for certain arithmetic functions,* J. of Mountain Res. **18**, No. 1 (2023) 1-10.
3. R. Sivaraman, J. D. Bulnes, J. López-Bonilla, *Complete Bell polynomials and recurrence relations for arithmetic functions,* European J. of Theor. Appl. Sci. **1**, No. 3 (2023) 51-55.
4. T. M. Apostol, *Introduction to analytic number theory,* Springer-Verlag, New York (1976).
5. G. H. Hardy, E. M. Wright, *An introduction to the theory of numbers,* Clarendon Press, Oxford (1979).
6. R. Sivaramakrishnan, *Classical theory of arithmetic functions,* Marcel Dekker, New York (1989).
7. R. Sivaraman, J. López-Bonilla, *Apostol-Robbins theorem applied to several arithmetic functions,* Bull. Math. Stat. & Res. **11**, No. 2 (2023) 82-84.
8. R. Sivaraman, J. D. Bulnes, J. López-Bonilla, *Sum of divisors function,* Int. J. of Maths. and Computer Res. **11**, No. 7 (2023) 3540-3542.
9. O. Lazarev, M. Mizuhara, B. Reid, *Some results in partitions, plane partitions, and multipartitions,* Summer 2010 REU Program in Maths. at Oregon State University, Aug 13, 2010.
10. J. López-Bonilla, J. Yaljá Montiel-Pérez, *On the Gandhi's recurrence relation for colour partitions,* Scientific Res. J. of Multidisciplinary **1**, No. 1 (2021) 24-25.
11. J. López-Bonilla, A. Lucas-Bravo, O. Marin-Martínez, *On the colour partitions* $p_r(n)$,Comput. Appl. Math. Sci. **6**, No. 2 (2021) 35-37.
12. J. López-Bonilla, M. Morales-García, *Sum of divisors function in terms of colour partitions,* Studies in Nonlinear Sci. **7**, No. 1 (2022) 4-5.
13. K. S. Kölbig, *The complete Bell polynomials for certain arguments in terms of Stirling numbers of the first kind,* J. Comput. Appl. Math. **51** (1994) 113-116.
14. W. P. Johnson, *The curious history of Faä di Bruno's formula,* The Math. Assoc. of America **109** (2002) 217-234.
15. D. F. Connon, *Various applications of the (exponential) complete Bell polynomials,* http://arxiv.org/ftp/arkiv/papers/1001/1001.2835.pdf , 16 Jan 2010.
16. J. López-Bonilla, S. Vidal-Beltrán, A. Zúñiga-Segundo, *Some applications of complete Bell polynomials,* World Eng. & Appl. Sci. J. **9**, No. 3 (2018) 89-92.
17. J. López-Bonilla, R. López-Vázquez, S. Vidal-Beltrán, *Bell polynomials,* Prespacetime J. **9**, No. 5 (2018) 451-453.
18. J. López-Bonilla, S. Vidal-Beltrán, A. Zúñiga-Segundo, *Characteristic equation of a matrix via Bell polynomials,* Asia Mathematika **2**, No. 2 (2018) 49-51.
19. https://en.wikipedia.org/wiki/Bell_polynomials#Definitions
20. S. Ramanujan, *On certain arithmetical functions,* Trans. Camb. Phil. Soc. **22**, No. 9 (1916) 159-184.
21. http://oeis.org
22. D. H. Lehmer, *Ramanujan's function*$\tau(n)$, Duke Math. J. **10** (1943) 483-492.
23. B. Jordan, B. Kelly, *The vanishing of the Ramanujan tau function,* Preprint, 12 March (1999).
24. http://mathworld.wolfram.com/TauFunction.html
25. D. Niebur, *A formula for Ramanujan's tau function,* Illinois J. Math. **19**, No. 3 (1975) 448-449.
26. http://en.wikipedia.org/wiki/Ramanujan_tau_function
27. T. M. Apostol, *Modular functions and Dirichlet series in number theory,* Springer-Verlag, New York (1997).
28. J. M. Gandhi, *Congruences for* $p_r(n)$*and Ramanujan's* τ*-function,* Amer. Math. Monthly **70**, No. 3 (1963) 265-274.
29. M. S. Devi* et al., "Linear Attribute Projection and Performance Assessment for Signifying the Absenteeism at Work using Machine Learning," International Journal of Recent Technology and Engineering (IJRTE), vol. 8, no. 3. Blue Eyes Intelligence Engineering and Sciences Engineering and Sciences Publication - BEIESP, pp. 1262–1267, Sep. 30, 2019. doi: 10.35940/ijrte.c4405.098319. Available: http://dx.doi.org/10.35940/ijrte.C4405.098319
30. P. Venkatasubbu and M. Ganesh, "Used Cars Price Prediction using Supervised Learning Techniques," International Journal of Engineering and Advanced Technology, vol. 9, no. 1s3. Blue Eyes Intelligence Engineering and Sciences Engineering and Sciences Publication - BEIESP, pp. 216–223, Dec. 31, 2019. doi: 10.35940/ijeat.a1042.1291s319. Available: http://dx.doi.org/10.35940/ijeat.A1042.1291S319
31. Dr. S. Bhubaneswari N T* and Dr. S. J, "A Research on the Factors that Impact the Effectiveness of Organizations in IT AND ITES in Coimbatore," International Journal of Innovative Technology and Exploring Engineering, vol. 8, no. 11. Blue Eyes Intelligence Engineering and Sciences Engineering and Sciences Publication - BEIESP, pp. 1011–1013, Sep. 30, 2019. doi: 10.35940/ijitee.i8070.0981119. Available: http://dx.doi.org/10.35940/ijitee.I8070.0981119

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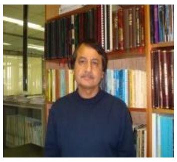

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